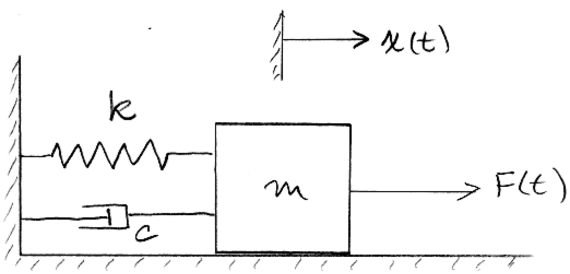
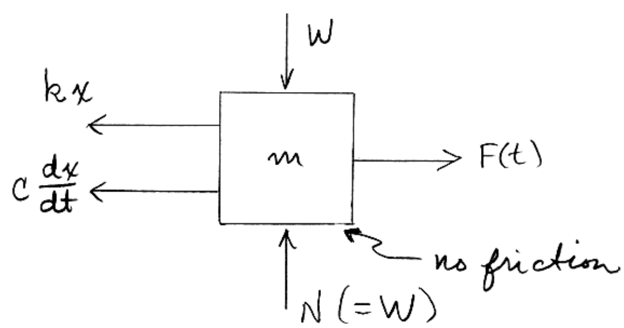
ME 2580 Dynamics
Introduction to Vibrations

Vibrations are a very common problem in mechanical and structural systems. Vibrations can lead to problems with wear, material fatigue, and noise that can affect the performance of the system. As an introduction to this important topic, consider the following simple spring-mass-damper (SMD) system shown below.

The mass m is excited by the external force $F(t)$ and the linear spring and damper that are connected to the wall. The displacement $x(t)$ is measured from the position where the spring is ***unstretched*** (known as the equilibrium position). When the mass is away from its equilibrium position, the linear spring applies a force proportional to the displacement ($x(t)$) and the linear damper applies a force proportional to the velocity ($\dot{x}(t)$) as shown in the free body diagram.





Summing forces in the direction of $x(t)$ gives

$$\sum F_x = F(t) - kx - c\frac{dx}{dt} = ma_x = m\frac{d^2x}{dt^2}$$

or

$$\boxed{\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \frac{F(t)}{m}}$$

This last equation is a linear, second order, constant coefficient, ordinary differential equation. It is the equation of motion of the mass and describes the motion of the mass for all time. Why?

Free, Undamped Response ($F(t)=0,\ c=0$)

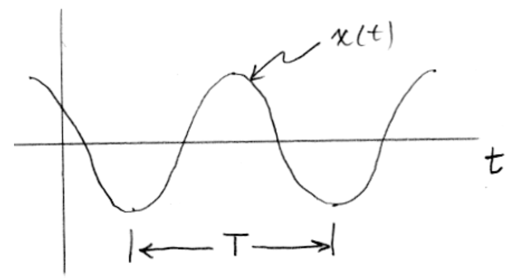
In the case of free, undamped response the differential equation of motion becomes

$$\boxed{\ddot{x}+\left(\frac{k}{m}\right)x=0}$$

The ***solution*** to this differential equation is $\boxed{x(t)=A\sin(\omega_n t-\phi)}$ where $\boxed{\omega_n=\sqrt{k/m}\ \text{(rad/s)}}$ is called the ***natural frequency*** of the system. So, if the mass is moved to the right and released from rest, it will oscillate about the equilibrium position with frequency ω_n. The values of A and ϕ depend on the initial conditions.

The natural frequency is often given in cycles/second or Hertz (Hz). This is found from ω_n to be

$$\boxed{f=\omega_n/2\pi}.$$



The ***period*** of oscillation T is defined as the number of seconds required to complete one cycle, so $\boxed{T=1/f}$.

Free, Damped Response ($F(t)=0$)

For the case of free, damped response, the differential equation of motion becomes

$$\ddot{x}+\left(\frac{c}{m}\right)\dot{x}+\left(\frac{k}{m}\right)x=0$$

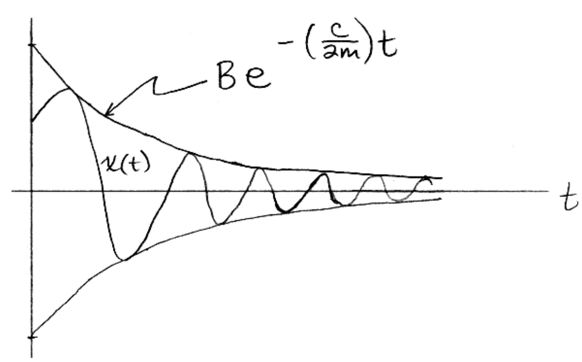
To characterize the motion of this type of system, define the ***damping ratio*** $\zeta=\dfrac{c}{2\sqrt{mk}}$. This represents the ratio of the actual damping factor c to the "***critical***" damping factor c_{critical}. In this way, the response can be characterized by the value of ζ as shown in the following table.

Value	Type of Response	Description of Response
$0<\zeta<1$	Under-damped	system oscillations decay with time
$\zeta=1$	Critically Damped	system response decays with time, no oscillation
$\zeta>1$	Over-damped	system response decays with time, no oscillation

Case1: Under-damped Response ($0<\zeta<1$)

For under-damped systems, the solution to the differential equation of motion is

$$x(t)=B\,e^{(-c/2m)\,t}\sin(\omega_d\,t+\phi)$$



where, as before, the values of B and ϕ depend on the initial conditions. The damped frequency of oscillation is

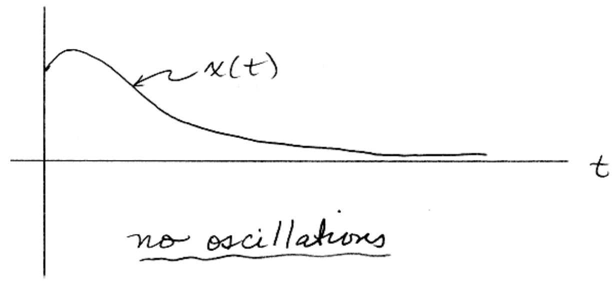
$$\omega_d=\sqrt{\omega_n^2-(c/2m)^2}\,.$$

Case 2: Critically Damped Response ($\zeta = 1$)

For ***critically damped*** systems, the solution to the differential equation of motion is

$$\boxed{x(t) = A\,e^{-at} + B\,t\,e^{-bt}}$$

where the values of A and B depend on the ***initial conditions***. Note that in this case there is just enough damping to stop the oscillations from occurring. A plot of this type of response is shown at the right.

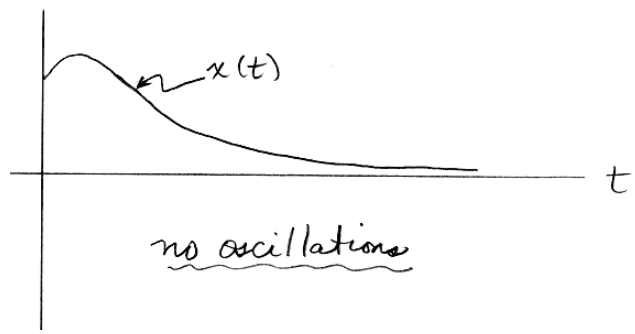


Case 3: Over-damped Response ($\zeta > 1$)

For ***over-damped*** systems, the solution to the differential equation of motion is

$$\boxed{x(t) = A\,e^{-a\,t} + B\,e^{-b\,t}}$$

where the values of A and B depend on the ***initial conditions***. Note that in this case there is more than enough damping to stop the oscillations from occurring.



Forced, Damped Response

In this case, the differential equation of motion is

$$\boxed{\ddot{x}+\left(\frac{c}{m}\right)\dot{x}+\left(\frac{k}{m}\right)x=\frac{F(t)}{m}}.$$

The ***solution*** of this differential equation is of the form

$$\boxed{x(t)=x_H(t)+x_P(t)}$$

where $x_H(t)$ represents the "***homogeneous***" solution (i.e. the free response when $F(t)=0$), and $x_P(t)$ represents the "***particular***" solution. The particular solution has the same form as the forcing function $F(t)$.

Definitions

Transient Response:	The part of $x(t)$ that decays with time.
Steady-State Response:	The part of $x(t)$ that remains after all decaying parts have vanished.