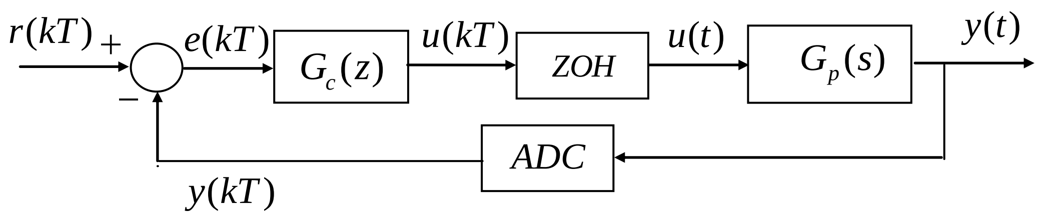
ME 4710 Motion and Control
Continuous and Equivalent Discrete Transfer Functions

- The closed loop control system shown below has a ***digital compensator*** represented by the discrete transfer function $G_c(z)$ and a ***continuous plant*** represented by the continuous transfer function $G_p(s)$. The digital control signal $(u(kt))$ is shown to be converted into a continuous signal $u(t)$ using a ***zero-order hold*** (ZOH).



Block Diagram of a Continuous/Discrete Closed-Loop System

- One common method of finding $G_c(z)$ involves a two step process.
  - First, design its continuous counterpart $G_c(s)$ using continuous compensator design techniques (root locus and /or Bode diagrams)
  - Then transform the resulting transfer function into a discrete form

  Some authors refer to this as ***emulation***, because the discrete compensator emulates its continuous counterpart.

- ***Tustin's approximation*** is one common method used to make this transformation. In this method, the substitution $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$ is made into the continuous transfer function to find $G_c(z)$.

- For example, the continuous phase lead compensator

$$\boxed{G_c(s) = \frac{3(s+5)}{s+15}}$$

  becomes

$$G_c(z) = \frac{3\left(\frac{2}{T}\left(\frac{z-1}{z+1}\right)+5\right)}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+15} = \frac{3\left(\frac{2}{T}(z-1)+5(z+1)\right)}{\frac{2}{T}(z-1)+15(z+1)} = \frac{\left(15+\frac{6}{T}\right)z+\left(15-\frac{6}{T}\right)}{\left(15+\frac{2}{T}\right)z+\left(15-\frac{2}{T}\right)}.$$

- For a sample time $T = 0.001$ (sec), we get

$$\boxed{G_c(z) = \frac{6015z - 5985}{2015z - 1985} = \frac{2.9851z - 2.97}{z - 0.9851}}$$

- It can be shown that when Tustin's approximation is applied to an ***integral compensator***, the ***trapezoidal rule*** is used to approximate the integral.

- MATLAB's "***c2d***" command can also be used to convert from ***continuous*** to ***discrete*** transfer functions. The methods of conversion include the ***Tustin*** approximation and a ***zero-order hold*** on the input to the transfer function.

```
>> num = 3*[1,5];
>> den = [1,15];
>> sys = tf(num,den)

    Transfer function:
        3 s + 15
        --------
        s + 15

>> sysD = c2d(sys,0.001,'tustin')

    Transfer function:
        2.985 z - 2.97
        --------------
        z - 0.9851

    Sampling time: 0.001
```

- Given the ***discrete transfer function***, the compensator can be written as a ***difference equation*** as follows.

$$G_c(z) = \frac{U(z)}{E(z)} = \frac{6015z - 5985}{2015z - 1985} = \frac{2.985z - 2.97}{z - 0.9851}$$

or

$$(z - 0.9851)U(z) = (2.985z - 2.97)E(z)$$

Multiply by z^{-1} and solving for $U(z)$

$$(1 - 0.9851z^{-1})U(z) = (2.985 - 2.97z^{-1})E(z)$$

or

$$\boxed{U(z) = 0.9851z^{-1}U(z) + 2.985E(z) - 2.97z^{-1}E(z)}$$

- This last result is ***equivalent*** to the ***difference equation***

$$\boxed{u(k) = 0.9851\,u(k-1) + 2.985\,e(k) - 2.97\,e(k-1)}$$

- ***Another common method*** is the ***MPZ (matched pole-zero)*** Method
- As with ***any numerical method***, it provides an ***approximation*** of the original continuous transfer function. The accuracy of the approximation is usually ***application dependent***.
- The *MPZ* method is based on ***mapping*** the ***poles*** and ***zeros*** of the continuous transfer function using the relationship $z = e^{sT}$ and ***preserving*** the low frequency gain.
- For example, if we have a ***phase lead or lag type compensator*** of the form

$$\boxed{G_c(s) = K\left(\frac{s+a}{s+b}\right)}$$

then, the ***discrete equivalent*** for a given sample time T is

$$\boxed{G_c(z) = K'\left(\frac{z-e^{-aT}}{z-e^{-bT}}\right)}$$

- Here, K' is found by applying the ***final value theorems*** to each transfer function and equating the results.

$$\lim_{s\to 0}\left(\$\cdot\tfrac{1}{\$}\cdot G_c(s)\right) = K\left(\tfrac{a}{b}\right) = \lim_{z\to 1}\left(\left(\frac{1-z^{-1}}{1-z^{-1}}\right)\cdot G_c(z)\right) = K'\left(\frac{1-e^{-aT}}{1-e^{-bT}}\right)$$

or

$$\boxed{K' = K\left(\tfrac{a}{b}\right)\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)}$$

- ***More details*** on this method may be found in Franklin, Powell, and Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice-Hall, 2002.