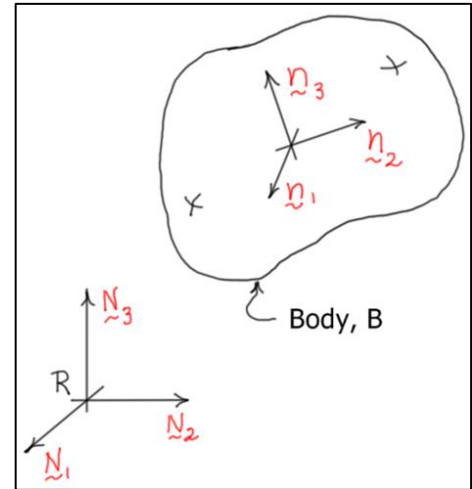


ME 6590 Multibody Dynamics

Orientation Angles of a Rigid Body in Three Dimensions

To describe the *general orientation* of a rigid body in three dimensions, consider the rigid body shown in the figure at the right. Here there are two reference frames – the base frame $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and the body-fixed frame $B: (\underline{n}_1, \underline{n}_2, \underline{n}_3)$. In any arbitrary position none of the unit vectors of the two frames are aligned. There are generally two methods for describing the orientation of B relative to the base frame R .



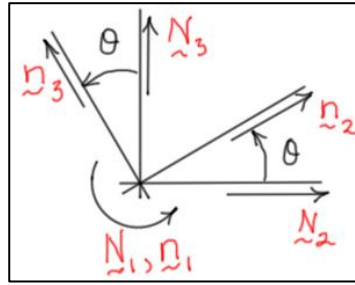
The first (and most commonly used) method of orienting a body in three dimensions involves the use of *orientation angles*. These are easy to visualize, but they are not unique, and they give rise to *singularities* in certain positions. The second method involves the use of *Euler* (or Euler-like) *parameters*. These are not easy to visualize; however, they are unique, and they have *no singularities*. The following notes discuss the use of orientation angles to describe angular position and motion of rigid bodies.

Simple Rotations

Simple rotations are defined as *right-handed* (or dextral) rotations about a single axis. For example, assume initially that the directions $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ are aligned with the directions $(\underline{N}_1, \underline{N}_2, \underline{N}_3)$. Then, an *X-rotation* is defined as a *right-handed* (or dextral) rotation of B about \underline{N}_1 (or \underline{n}_1), a *Y-rotation* as a *right-handed* rotation about \underline{N}_2 (or \underline{n}_2), and a *Z-rotation* as a *right-handed* rotation about \underline{N}_3 (or \underline{n}_3). For each of these simple rotations, the unit vectors of the two reference frames can be related to each other using matrix equations.

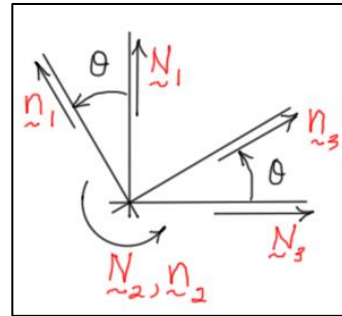
X-Rotation:

$$\begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$



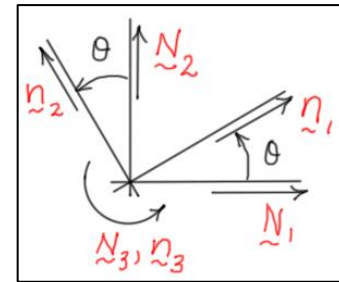
Y-Rotation:

$$\begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$



Z-Rotation:

$$\begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

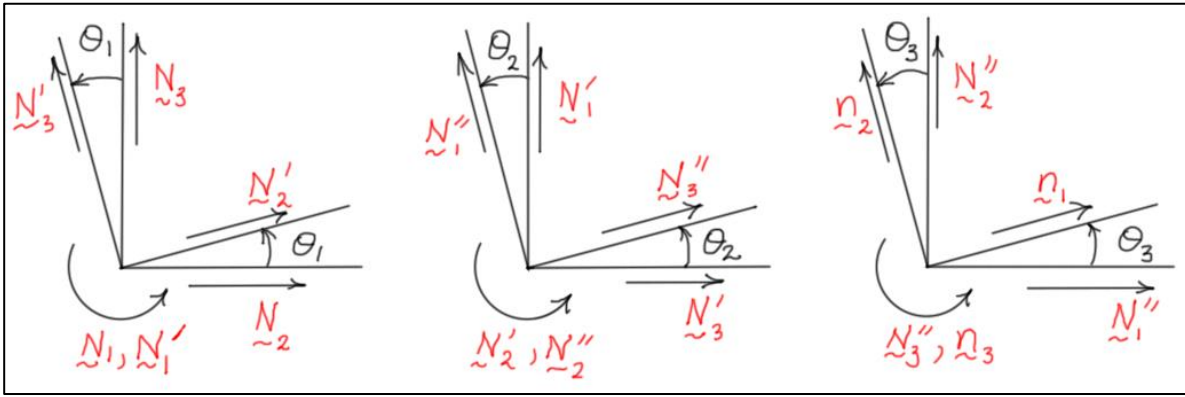


Here, S_θ and C_θ represent the *sine* and *cosine* of the rotation angle θ .

The coefficient matrices in the above equations are called “*transformation*” or “*rotation*” matrices. They are *orthogonal* matrices with a *determinant* of +1. As with all orthogonal matrices, the *inverses* of these matrices are simply their *transposes*. Hence, it is easy to invert the above equations.

General Orientations

A rigid body can be moved into any arbitrary orientation (relative to a base frame) using a sequence of three simple rotations. These rotations can occur about the base-frame axes or the body-frame axes. One common example is a *body-fixed* 1-2-3 rotation sequence. (Here, "1-2-3" has been used to stand for n_1, n_2, n_3 rotations.) To work through the sequence of rotations, *intermediate reference frames* may be introduced as shown below.



The **matrix equations** for the three rotations can be written as

$$\begin{Bmatrix} \tilde{N}'_1 \\ \tilde{N}'_2 \\ \tilde{N}'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & S_1 \\ 0 & -S_1 & C_1 \end{bmatrix} \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = [R_1] \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} \quad \begin{Bmatrix} \tilde{N}''_1 \\ \tilde{N}''_2 \\ \tilde{N}''_3 \end{Bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix} \begin{Bmatrix} \tilde{N}'_1 \\ \tilde{N}'_2 \\ \tilde{N}'_3 \end{Bmatrix} = [R_2] \begin{Bmatrix} \tilde{N}'_1 \\ \tilde{N}'_2 \\ \tilde{N}'_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{Bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{N}''_1 \\ \tilde{N}''_2 \\ \tilde{N}''_3 \end{Bmatrix} = [R_3] \begin{Bmatrix} \tilde{N}''_1 \\ \tilde{N}''_2 \\ \tilde{N}''_3 \end{Bmatrix}$$

As before, S_i and C_i represent the **sine** and **cosine** of the orientation angle θ_i .

These equations can be **combined** to form the matrix relationship between the base-fixed and the body-fixed unit vectors as follows

$$\begin{Bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{Bmatrix} = [R_3][R_2][R_1] \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \tilde{N}_1 \\ \tilde{N}_2 \\ \tilde{N}_3 \end{Bmatrix}$$

So, for a body-fixed 1-2-3 rotation sequence, the **transformation matrix** that relates the unit vectors in the body reference frame to those in the base reference frame is

$$[R] = [R_3][R_2][R_1] = \begin{bmatrix} C_2C_3 & C_1S_3 + S_1S_2C_3 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & C_1C_3 - S_1S_2S_3 & S_1C_3 + C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}$$

Here, the matrices $[R_i]$ are defined in the above equations. Like the individual rotation matrices $[R_i]$, the matrix $[R]$ is an *orthogonal matrix* whose *determinant* is +1. So, again it is easy to *invert* the relationship between the unit vector sets.

Note: Transformation matrices for many different combinations of rotations are given in Appendix I of the text *Spacecraft Dynamics* by T. R. Kane, P. W. Likins, and D. A. Levinson, McGraw-Hill, 1983.

Relationship Between Vector Components in the Base and Body Frames

Given a vector \underline{A} expressed in two different reference frames

$$\underline{A} = A_1 \underline{N}_1 + A_2 \underline{N}_2 + A_3 \underline{N}_3 = a_1 \underline{n}_1 + a_2 \underline{n}_2 + a_3 \underline{n}_3,$$

the vector components can be related using the transformation matrix as follows

$$\{A\} = [R]^T \{a\} \quad \text{or} \quad \{a\} = [R]\{A\}$$

Transformation Matrices and Direction Cosines

The elements of a transformation matrix that relates the unit vectors of two different reference frames are the *direction cosines* of the various unit vector pairs. Given that R_{ij} represents the elements of the transformation matrix $[R]$, then

$$R_{ij} = \underline{n}_i \cdot \underline{N}_j = C_{\theta_{ij}}$$

Here, $C_{\theta_{ij}}$ represents the *cosine* of the angle between the unit vectors \underline{n}_i and \underline{N}_j .

