ME 6590 Multibody Dynamics
Coordinate Transformation (Rotation) Matrices

Relationships Between Unit Vectors in Different Reference Frames

The unit vectors of two mutually perpendicular unit vector sets $A:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ and $B:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ can be related using transformation matrices. To do this, write

$$\boxed{\underset{\sim}{e}_i=\sum_{j=1}^{3}\left(\underset{\sim}{e}_i\cdot\underset{\sim}{n}_j\right)\underset{\sim}{n}_j}\quad (i=1,2,3)$$

Or, in matrix form,

$$\{e\}=\begin{Bmatrix}\underset{\sim}{e}_1\\ \underset{\sim}{e}_2\\ \underset{\sim}{e}_3\end{Bmatrix}=\begin{bmatrix}\left(\underset{\sim}{e}_1\cdot\underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_1\cdot\underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_1\cdot\underset{\sim}{n}_3\right)\\ \left(\underset{\sim}{e}_2\cdot\underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_2\cdot\underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_2\cdot\underset{\sim}{n}_3\right)\\ \left(\underset{\sim}{e}_3\cdot\underset{\sim}{n}_1\right) & \left(\underset{\sim}{e}_3\cdot\underset{\sim}{n}_2\right) & \left(\underset{\sim}{e}_3\cdot\underset{\sim}{n}_3\right)\end{bmatrix}\begin{Bmatrix}\underset{\sim}{n}_1\\ \underset{\sim}{n}_2\\ \underset{\sim}{n}_3\end{Bmatrix}=\begin{bmatrix}R_{11} & R_{12} & R_{13}\\ R_{21} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{n}_1\\ \underset{\sim}{n}_2\\ \underset{\sim}{n}_3\end{Bmatrix}=[R]\{n\}$$

where $\boxed{R_{ij}=\underset{\sim}{e}_i\cdot\underset{\sim}{n}_j=\cos\left(\underset{\sim}{e}_i,\underset{\sim}{n}_j\right)}$ represents the ***cosine*** of the angle between the unit vectors $\underset{\sim}{e}_i$ and $\underset{\sim}{n}_j$. The 3×3 matrix $[R]$ is called the ***direction cosine matrix***. It can be shown that the matrix $[R]$ is ***orthogonal***, so its ***inverse*** is equal to its ***transpose***. Hence,

$\boxed{\{e\}=[R]\{n\}}$ and $\boxed{\{n\}=[R]^T\{e\}}$

Relationships Between Vector Components in Different Reference Frames

Given the representations of a vector $\underset{\sim}{a}$ in two different reference frames

$$\boxed{\underset{\sim}{a}=\sum_{i=1}^{3}a_i\,\underset{\sim}{n}_i=\sum_{i=1}^{3}a_i'\,\underset{\sim}{e}_i},$$

the components a_i $(i=1,2,3)$ can be related to the a_i' $(i=1,2,3)$ using transformation matrices as follows. Writing the above equation in matrix form

$$\boxed{\{a\}^T\{n\}=\{a'\}^T\{e\}=\{a'\}^T[R]\{n\}}$$

Comparing both sides of the equation, gives

$$\{a\}^T = \{a'\}^T [R] \quad \text{or} \quad \boxed{\{a\} = \left(\{a'\}^T [R]\right)^T = [R]^T \{a'\}}$$

Finally, using the fact that $[R]$ is ***orthogonal***

$$\boxed{\{a'\} = [R]\{a\}}$$

Dot and Cross Products Revisited

Transformation matrices can also be used to take the ***dot*** or ***cross*** products of two vectors expressed in two ***different*** reference frames as follows

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \quad \rightarrow \quad \boxed{\{a\}^T \{b\} = \{a\}^T [R]^T \{b'\}}$$

$$\underset{\sim}{a} \times \underset{\sim}{b} \quad \rightarrow \quad \boxed{[\tilde{a}]\{b\} = [\tilde{a}][R]^T \{b'\}}$$

Recall that $[\tilde{a}]$ is the skew-symmetric matrix defined as

$$\boxed{[\tilde{a}] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}$$