An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 1

Angular Velocity and Angular Acceleration: An Introduction

Summary

This unit introduces the concepts of ***angular velocity*** and ***angular acceleration*** vectors and shows ***how to calculate*** them for mechanical systems in which components are connected by simple revolute (pin) joints. These ***concepts*** will be ***generalized*** in Unit 5 to apply to systems with more ***complex connecting joints***.

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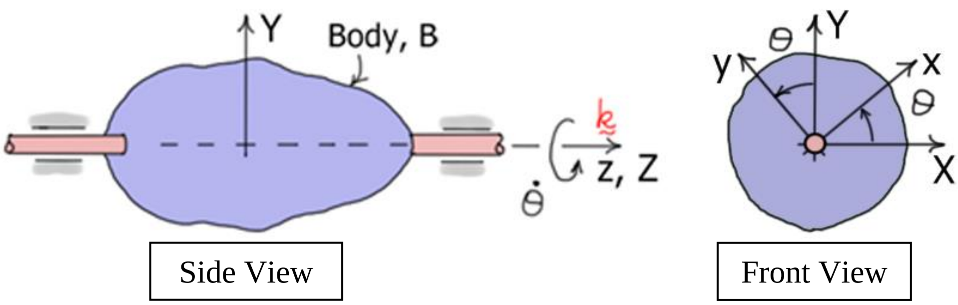
Simple Rotational Motion

Simple Angular Velocity

The rigid body B shown in the diagram below rotates about the Z-axis. The XYZ reference frame is a ***fixed*** (non-rotating) frame, while the xyz reference frame is fixed-in and ***rotates*** with the body B. Angle θ is defined as the ***angle of rotation*** of B, and $\dot{\theta}$ is the ***rotation rate***.



The ***angular velocity*** of B in R (written here as ${}^{R}\underset{\sim}{\omega}_{B}$) is defined as

$$ {}^{R}\underset{\sim}{\omega}_{B} = \frac{d\theta}{dt}\,\underset{\sim}{k} = \dot{\theta}\,\underset{\sim}{k} $$

The ***magnitude*** of ${}^{R}\underset{\sim}{\omega}_{B}$ is the ***rate*** of rotation (usually expressed in radians/second), and its ***direction*** is defined by the "***right-hand***" rule. Let the fingers of your right hand point in the direction of the rotation, and your thumb points in the direction of ${}^{R}\underset{\sim}{\omega}_{B}$. Note that time derivatives of scalar functions (e.g. $\theta(t)$) are often indicated using a " $\cdot$ " over the function name (e.g. $\dot{\theta}(t)$).

Simple Angular Acceleration

The ***angular acceleration*** of B in R (written as ${}^{R}\underset{\sim}{\alpha}_{B}$) is found by ***differentiating*** the ***angular velocity*** vector. That is,

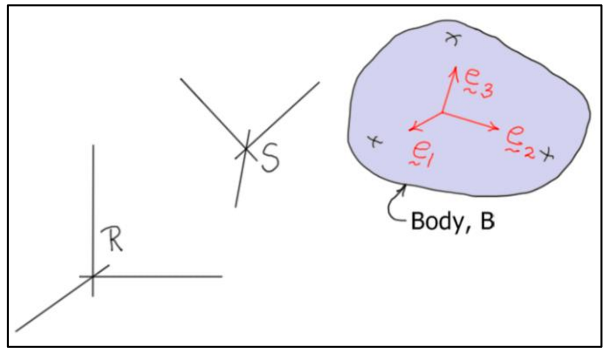
$$ {}^{R}\underset{\sim}{\alpha}_{B} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_{B}\right) = \ddot{\theta}\,\underset{\sim}{k} $$

Here, $\ddot{\theta}$ represents $\dfrac{d^2\theta}{dt^2}$ and is usually expressed in units of radians/second^2. Also, note that the ***derivative*** of ${}^{R}\underset{\sim}{\omega}_{B}$ is taken in the reference frame R so unit vectors fixed in R are taken as constant.

Complex Rotational Motion

Angular Velocity: Summation Rule

Consider a rigid body *B* undergoing three dimensional motion as shown in the diagram below. *R* and *S* represent ***two reference frames rotating*** relative to each other. The ***angular velocity*** of the body *B* relative to the reference frame *R* (again, written as ${}^R\underset{\sim}{\omega}_B$) may be found by using the ***summation rule for angular velocities*** to work through the intermediate reference frame *S*.

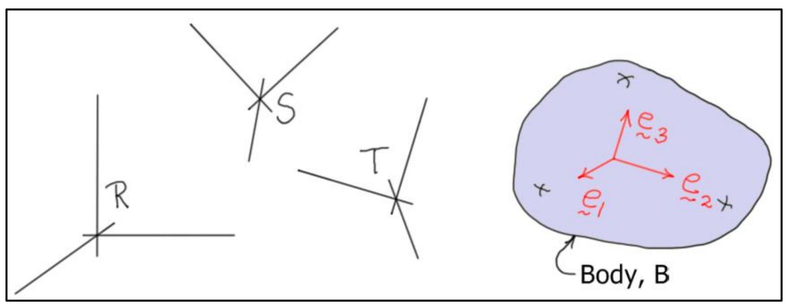


$$\boxed{{}^R\underset{\sim}{\omega}_B = {}^S\underset{\sim}{\omega}_B + {}^R\underset{\sim}{\omega}_S}$$

Here, ${}^S\underset{\sim}{\omega}_B$ represents the ***angular velocity*** of *B* relative to the reference frame *S*, and ${}^R\underset{\sim}{\omega}_S$ represents the ***angular velocity*** of frame *S* relative to *R*.

Consider next the body *B* in the the diagram below. Here, there are ***three*** reference frames (*R*, *S*, and *T*) ***all rotating*** relative to each other. In this case, ${}^R\underset{\sim}{\omega}_B$ the angular velocity of *B* relative to *R* may be found using the ***summation rule*** to work through the intermediate frames *S* and *T*.

$$\boxed{\begin{aligned} {}^R\underset{\sim}{\omega}_B &= {}^T\underset{\sim}{\omega}_B + {}^R\underset{\sim}{\omega}_T \\ &= {}^T\underset{\sim}{\omega}_B + {}^S\underset{\sim}{\omega}_T + {}^R\underset{\sim}{\omega}_S \end{aligned}}$$



In fact, this rule may be ***extended*** to as many reference frames as necessary.

The ***summation rule*** may be used to compute the ***angular velocity*** of a body (undergoing three-dimensional motion) by introducing a set of ***reference frames*** whose relative angular motions may be described using ***simple angular velocities***. Then, the angular velocity of the body is found by ***summing*** the simple angular velocities.

Angular Acceleration

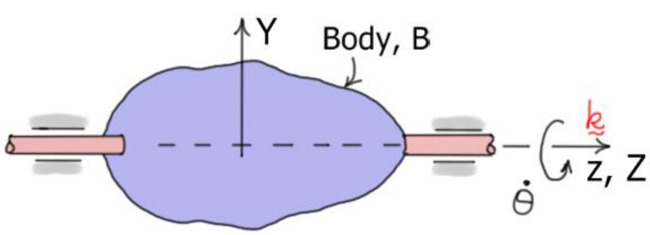
There is ***no*** corresponding summation rule for ***angular acceleration***. As with simple angular motion, the ***angular acceleration*** of a body is found by ***direct differentiation*** of the ***angular velocity*** vector.

$$\boxed{{}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right)}$$

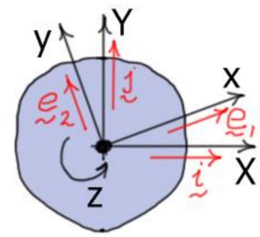
Differentiating Unit Vectors Using the Angular Velocity Vector

Simple Rotational Motion

Consider again a rigid body *B* rotating about a single axis. As before, the *XYZ* reference frame is a ***fixed*** frame, while the *xyz* reference frame is fixed-in and ***rotates*** with the body. Here, the directions of the *XYZ* reference frame are represented using the ***unit vector set*** $R:(\underset{\sim}{i},\ \underset{\sim}{j},\underset{\sim}{k})$, and the directions of the *xyz* reference frame using the unit vector set $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{k})$. Note that each unit vector set is a ***right-handed*** set, that is $\underset{\sim}{i}\times\underset{\sim}{j}=\underset{\sim}{k}$ and $\underset{\sim}{e}_1\times\underset{\sim}{e}_2=\underset{\sim}{k}$.



Side View



Front View

Unit vectors fixed in *B* can be ***differentiated*** using the concept of ***angular velocity***. It can be shown that

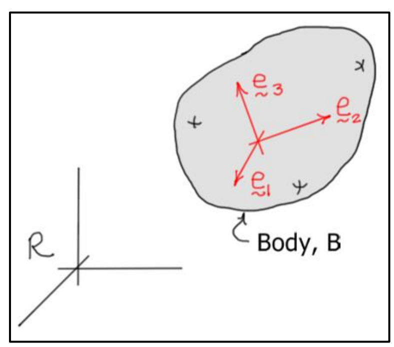
$$\boxed{\frac{{}^R d\underset{\sim}{e}_i}{dt}={}^R\underset{\sim}{\omega}_B\times\underset{\sim}{e}_i}\qquad (i=1,2)$$

Here $\frac{{}^R d\underset{\sim}{e}_i}{dt}$ represents the ***derivative*** of the unit vector $\underset{\sim}{e}_i$ ***in reference frame*** R.

<u>Aside:</u> $\boxed{\frac{{}^R d\underset{\sim}{e}_1}{dt}=\frac{{}^R d}{dt}(C_\theta\underset{\sim}{i}+S_\theta\underset{\sim}{j})=\dot{\theta}(-S_\theta\underset{\sim}{i}+C_\theta\underset{\sim}{j})\ =\ \dot{\theta}\underset{\sim}{e}_2=\dot{\theta}(\underset{\sim}{k}\times\underset{\sim}{e}_1)\ =\ {}^R\underset{\sim}{\omega}_B\times\underset{\sim}{e}_1}$

Note here that S_θ and C_θ have been used to represent the sine and cosine of angle θ.

<u>Differentiation of Unit Vectors – The General Case</u>



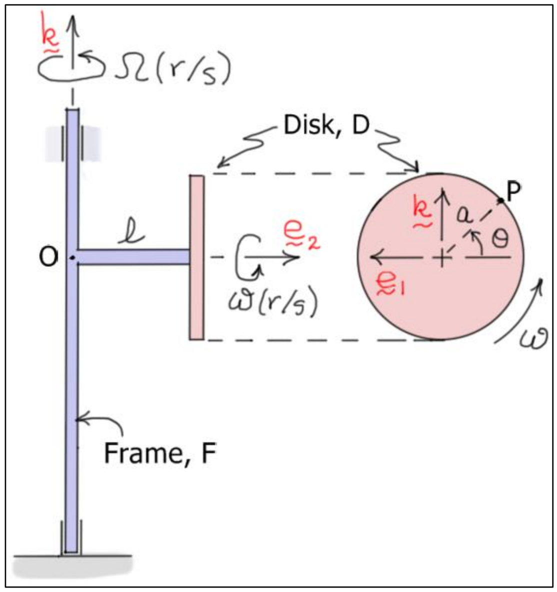
Consider now a rigid body *B* moving in ***three-dimensional space***. Given a set of unit vectors $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ fixed in *B*, it can be shown that

$$\boxed{\frac{{}^R d\underset{\sim}{e}_i}{dt}={}^R\underset{\sim}{\omega}_B\times\underset{\sim}{e}_i}\qquad (i=1,2,3)$$

As before, $\frac{{}^R d\underset{\sim}{e}_i}{dt}$ represents the ***derivative*** of unit vector $\underset{\sim}{e}_i$ in the reference frame R, and ${}^R\underset{\sim}{\omega}_B$ is the ***angular velocity*** of *B* in *R*.

Example 1:

The system shown consists of two connected bodies – the frame F and the disk D. Frame F rotates at a rate of Ω (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed-to and rotates relative to F at a rate of ω (rad/s) about the horizontal arm of F (annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames:

$R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ (fixed frame)

$F:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating frame)

Find: (express the results using unit vectors fixed in F)

a) ${}^R\underset{\sim}{\omega}_D$ … the ***angular velocity*** of disk D in R

b) ${}^R\underset{\sim}{\alpha}_D$ … the ***angular acceleration*** of disk D in R

Solution:

a) Using the summation rule: $\boxed{{}^R\underset{\sim}{\omega}_D = {}^F\underset{\sim}{\omega}_D + {}^R\underset{\sim}{\omega}_F = \omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}}$ (rad/s)

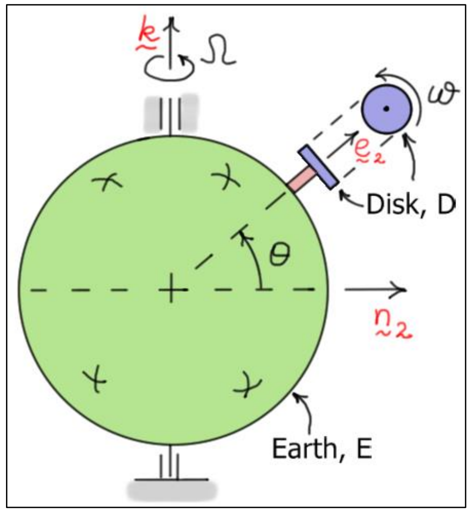
b) The angular acceleration is found by ***direct differentiation***.

$$
\begin{aligned}
{}^R\underset{\sim}{\alpha}_D &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_D\right) = \dot{\omega}\,\underset{\sim}{e}_2 + \omega\frac{{}^R d}{dt}\left(\underset{\sim}{e}_2\right) + \dot{\Omega}\,\underset{\sim}{k} + \Omega\underbrace{\frac{{}^R d}{dt}\left(\underset{\sim}{k}\right)}_{\text{zero}} \\
&= \dot{\omega}\,\underset{\sim}{e}_2 + \omega\left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{e}_2\right) + \dot{\Omega}\,\underset{\sim}{k} = \dot{\omega}\,\underset{\sim}{e}_2 + \omega\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{e}_2\right) + \dot{\Omega}\,\underset{\sim}{k}
\end{aligned}
$$

So,

$$\boxed{{}^R\underset{\sim}{\alpha}_D = -\omega\,\Omega\,\underset{\sim}{e}_1 + \dot{\omega}\,\underset{\sim}{e}_2 + \dot{\Omega}\,\underset{\sim}{k}} \quad (\text{rad/s}^2)$$

Example 2:



The system shown consists of two connected bodies – the earth E and the disk D. The earth is assumed to rotate at a rate Ω about a fixed direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed-to and rotates relative to E at a rate ω about a vertical axis at a latitude angle of θ (annotated by the unit vector $\underset{\sim}{e}_2$). The unit vector $\underset{\sim}{n}_2$ points outward along the equatorial plane, and $\underset{\sim}{e}_2$ is in the plane formed by the unit vectors $\underset{\sim}{n}_2$ and $\underset{\sim}{k}$.

Given:

$$\Omega = 1 \text{ (rev/day)} = 7.2722 \times 10^{-5} \text{ (rad/s)} = \text{constant}$$
$$\omega = 10{,}000 \text{ (rpm)} = 1047.2 \text{ (rad/s)} = \text{constant}$$

Reference frame:

$$E:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{k}) \text{ (rotating frame)} \quad \left(\underset{\sim}{n}_2 \times \underset{\sim}{k} = \underset{\sim}{n}_1\right)$$

Find: (express the results using unit vectors fixed in E)

a) ${}^R\underset{\sim}{\omega}_D$ … the ***angular velocity*** of disk D relative to a fixed frame

b) ${}^R\underset{\sim}{\alpha}_D$ … the ***angular acceleration*** of disk D relative to a fixed frame

Solution:

a) Using the summation rule:

$$ {}^R\underset{\sim}{\omega}_D = {}^E\underset{\sim}{\omega}_D + {}^R\underset{\sim}{\omega}_E = \omega\, \underset{\sim}{e}_2 + \Omega\, \underset{\sim}{k} = \omega \underbrace{\left(C_\theta\, \underset{\sim}{n}_2 + S_\theta\, \underset{\sim}{k}\right)}_{\underset{\sim}{e}_2} + \Omega\, \underset{\sim}{k} $$

$$ \boxed{ {}^R\underset{\sim}{\omega}_D = \left(\omega C_\theta\right) \underset{\sim}{n}_2 + \left(\omega S_\theta + \Omega\right) \underset{\sim}{k} } \text{ (rad/s)} $$

b) The angular acceleration is found by direct differentiation.

$$ {}^R\underset{\sim}{\alpha}_D = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_D\right) = \underbrace{\dot{\omega}\, \underset{\sim}{e}_2}_{\text{zero}} + \omega \frac{{}^R d}{dt}\left(\underset{\sim}{e}_2\right) + \underbrace{\dot{\Omega}\, \underset{\sim}{k}}_{\text{zero}} + \Omega \underbrace{\frac{{}^R d}{dt}\left(\underset{\sim}{k}\right)}_{\text{zero}} $$

$$ = \omega\left({}^R\underset{\sim}{\omega}_E \times \underset{\sim}{e}_2\right) = \omega\left(\Omega\, \underset{\sim}{k} \times \underset{\sim}{e}_2\right) = \omega\Omega\; \underset{\sim}{k} \times \left(C_\theta\, \underset{\sim}{n}_2 + S_\theta\, \underset{\sim}{k}\right) $$

So, a disk rotating at a constant rate of $10{,}000 \text{ (rpm)}$ at a latitude of $\theta = 40 \text{ (deg)}$ has an ***angular acceleration***

$$ \boxed{ {}^R\underset{\sim}{\alpha}_D = -\omega\, \Omega\, C_\theta\, \underset{\sim}{n}_1 = -\,0.0583\, \underset{\sim}{n}_1 } \text{ (rad/s}^2\text{)} \qquad \ldots\text{due to the Earth's rotation} $$

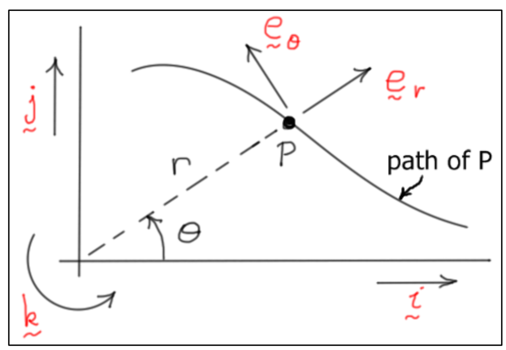
As above, S_θ and C_θ have been used to represent the ***sine*** and ***cosine*** of angle θ.

Notes:

1. Vector results can be expressed using ***any convenient set*** of ***unit vectors***. In the examples above, the results could also have been expressed using non-rotating unit vectors or unit vectors fixed in *D*. It is often convenient to express ${}^{R}\underset{\sim}{\omega}_{D}$ and ${}^{R}\underset{\sim}{\alpha}_{D}$ using unit vectors fixed in *D*. See Unit 5 for more details.
2. The angular velocities ${}^{R}\underset{\sim}{\omega}_{F}$ and ${}^{F}\underset{\sim}{\omega}_{D}$ in Example 1 and the angular velocities ${}^{R}\underset{\sim}{\omega}_{E}$ and ${}^{E}\underset{\sim}{\omega}_{D}$ in Example 2 are all ***simple angular velocities***. The angular velocities ${}^{R}\underset{\sim}{\omega}_{D}$ are ***not***. The summation rule enables us build ***complex angular velocities*** from a ***series of simple angular velocities***.
3. ***Differentiation*** of the angular velocity vector produces ***familiar*** terms from two dimensional analysis (such as "$\dot{\omega}\,\underset{\sim}{e}_2$" and "$\dot{\Omega}\,\underset{\sim}{k}$"), but it also produces ***less-familiar*** terms (such as "$\omega\Omega\,\underset{\sim}{e}_1$"). All of these terms are ***common*** in ***three-dimensional analysis*** of angular motion.
4. ***Derivatives*** of ***scalar functions*** (such as $\omega(t)$ and $\Omega(t)$) are ***independent*** of reference frames, whereas the ***derivatives*** of vectors depend on the reference frame in which the derivative is measured.
5. In this and subsequent units the notation "$B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$" will be used to indicate a right-handed set of unit vectors fixed in body *B*. The unit vectors are ordered so $\underset{\sim}{e}_1 \times \underset{\sim}{e}_2 = \underset{\sim}{e}_3$.

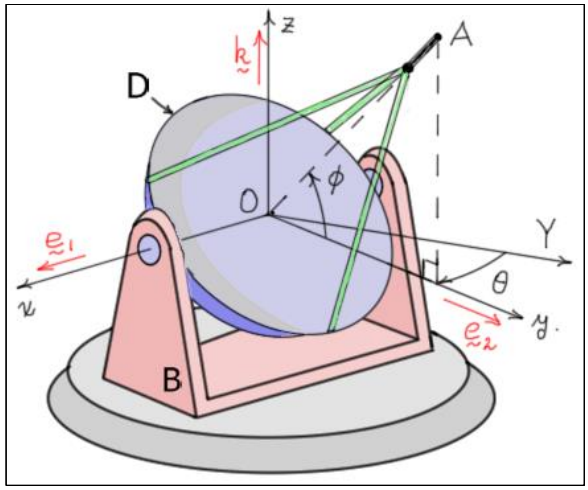
Exercises:

1.1 Radial & Transverse Components: The diagram shows two reference frames and a point *P* traveling along some path. The unit vector set $R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ defines directions in the fixed frame *R*, and the unit vector set $E:(\underset{\sim}{e}_r, \underset{\sim}{e}_\theta, \underset{\sim}{k})$ defines directions in a rotating frame *E*. Given that the angular velocity of *E* in *R* is ${}^{R}\underset{\sim}{\omega}_{E} = \dot{\theta}\,\underset{\sim}{k}$, find the time derivatives of the unit vectors $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_\theta$ in *R*.



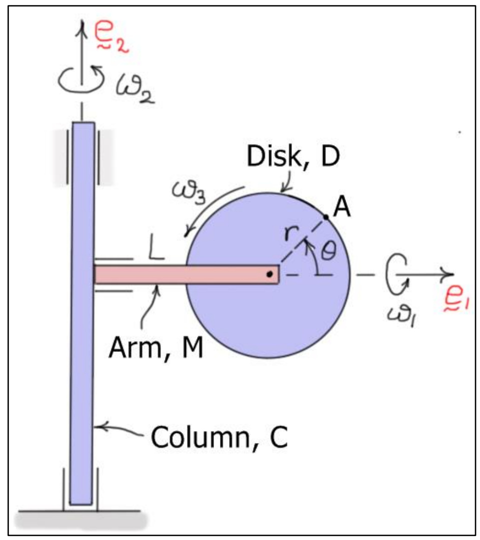
Answers: $\boxed{\dfrac{{}^{R}d\underset{\sim}{e}_r}{dt} = \dot{\theta}\,\underset{\sim}{e}_\theta}$ and $\boxed{\dfrac{{}^{R}d\underset{\sim}{e}_\theta}{dt} = -\dot{\theta}\,\underset{\sim}{e}_r}$

1.2 The antenna system shown has two components, the base *B* and the antenna dish *D*. Base *B* rotates relative to the ground about the fixed *z*-axis, and dish *D* rotates relative to *B* about the rotating *x*-axis. At any instant, the angle between the *y*-axis ($\underset{\sim}{e}_2$) and the fixed *Y*-axis is θ, and the angle between line *OA* and the *y*-axis is ϕ. Given values for θ, ϕ, and their time derivatives, find ${}^R\underset{\sim}{\omega}_D$ and ${}^R\underset{\sim}{\alpha}_D$ the angular velocity and angular acceleration of the dish *D* in a fixed reference frame *R*.



Answers: $\boxed{{}^R\underset{\sim}{\omega}_D = \dot{\phi}\,\underset{\sim}{e}_1 - \dot{\theta}\,\underset{\sim}{k}}$ and $\boxed{{}^R\underset{\sim}{\alpha}_D = \ddot{\phi}\,\underset{\sim}{e}_1 - \dot{\theta}\dot{\phi}\,\underset{\sim}{e}_2 - \ddot{\theta}\,\underset{\sim}{k}}$ (results expressed in $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$)

1.3 The system shown has three components, a vertical column *C*, a horizontal arm *M*, and a disk *D*. The disk rotates relative to the arm at a rate ω_3 (rad/sec) about the $\underset{\sim}{n}_3$ direction (normal to *D*), the arm rotates relative to the column at a rate of ω_1 (rad/sec) about the $\underset{\sim}{e}_1$ direction, and the column rotates relative to the ground at a rate of ω_2 (rad/sec) about the fixed $\underset{\sim}{e}_2$ direction. The unit vector set $C:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in the column, and the unit vector set $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ is fixed in arm M. Given values for ω_1, ω_2, ω_3, and their time derivatives, find ${}^R\underset{\sim}{\omega}_D$ and ${}^R\underset{\sim}{\alpha}_D$ the angular velocity and angular acceleration of *D* in the fixed reference frame *R*.



Answers: $\boxed{{}^R\underset{\sim}{\omega}_D = \omega_1\,\underset{\sim}{e}_1 + \omega_2 C_\phi\,\underset{\sim}{n}_2 + (\omega_3 - \omega_2 S_\phi)\,\underset{\sim}{n}_3}$ (results expressed in $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$)

$$\boxed{{}^R\underset{\sim}{\alpha}_D = (\dot{\omega}_1 + \omega_2\omega_3 C_\phi)\underset{\sim}{e}_1 + (\dot{\omega}_2 C_\phi - \omega_1\omega_2 S_\phi - \omega_1\omega_3)\underset{\sim}{n}_2 + (\dot{\omega}_3 - \dot{\omega}_2 S_\phi - \omega_1\omega_2 C_\phi)\underset{\sim}{n}_3}$$

<u>Hint</u>: Here ϕ is the angle between the plane of the disk and the $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ plane $\left(\dot{\phi} = \omega_1\right)$. The diagram shows the position where $\phi = 0$.

References:

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