Summary

This unit introduces the concepts of velocity and acceleration vectors and shows how to calculate them using direct differentiation. This technique can be applied to complex mechanical systems; however, at this point the focus will remain on systems in which components are connected by simple revolute (pin) joints. This technique will be generalized in Unit 7 to apply to systems with more complex connecting joints.

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Position, Velocity, and Acceleration

The diagram shows the three-dimensional motion of some point $P$ within a mechanical system relative to a reference frame $R$. Given $\mathbf{r}(t)$ the position vector of $P$ as a function of time, the velocity and acceleration of $P$ relative to $R$ are defined to be

$$
\mathbf{v}_P = \frac{d}{dt}(\mathbf{r}(t)) \quad \text{and} \quad \mathbf{a}_P = \frac{d}{dt}(\mathbf{v}_P)
$$

The velocity $\mathbf{v}_P$ is tangent to the path of $P$ at all times. The acceleration $\mathbf{a}_P$ generally has components tangent and normal to the path.

Using these fundamental definitions along with the concepts for differentiating unit vectors presented in Unit 1, the velocities and accelerations of a point within a mechanical system may be calculated by directly differentiating their position vectors. The following examples illustrate this process.

Example 1:

The system shown consists of two connected bodies – the frame $F$ and the disk $D$. Frame $F$ rotates at a rate of $\Omega$ (rad/s) about the fixed vertical direction (annotated by the unit vector $\mathbf{k}$). Disk $D$ is affixed to and rotates relative to $F$ at a rate of $\omega$ (rad/s) about the horizontal arm of $F$ (annotated by the rotating unit vector $\mathbf{e}_2$).

Reference frames:

$R$ : $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ (fixed frame)

$F$ : $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{k})$ (rotating frame)

Find: (express the results using unit vectors fixed in $F$)

a) $\mathbf{v}_P$ … the velocity of point $P$ in $R$ using direct differentiation

b) $\mathbf{a}_P$ … the acceleration of point $P$ in $R$ using direct differentiation
Solution:
a) First, construct a position vector for \( P \) that describes its position relative to \( R \) in an arbitrary (general) configuration. For example,

\[
\ell_{P/O} = \ell_{Q/O} + \ell_{P/Q} = \ell \mathbf{e}_2 + a \mathbf{e}_r
\]

Here, points \( O, P, \) and \( Q \) are shown in the diagram, and \( \mathbf{e}_r \) is a unit vector pointing from the center of the disk towards \( P \). Differentiating relative to the ground frame \( R \) gives

\[
\ddot{r}_P = \frac{d}{dt} \left( \ell \mathbf{e}_2 + a \mathbf{e}_r \right)
\]

\[
= \ell \frac{d}{dt} \left( \mathbf{e}_2 \right) + a \frac{d}{dt} \left( \mathbf{e}_r \right)
\]

\[
= \ell \left( \dot{\mathbf{r}}_P \times \mathbf{e}_2 \right) + a \left[ \dot{\mathbf{r}}_P \times \mathbf{e}_r \right]
\]

\[
= \ell \left( \mathbf{\Omega}_r \times \mathbf{k} \times \mathbf{e}_2 \right) + a \left[ \left( \mathbf{\omega}_r \times \mathbf{e}_2 + \mathbf{\Omega}_r \times \mathbf{k} \right) \times \mathbf{e}_r \right]
\]

\[
= -\ell \mathbf{\Omega}_r \mathbf{e}_1 + a \left[ \left( \mathbf{\omega}_r \times \mathbf{e}_2 + \mathbf{\Omega}_r \times \mathbf{k} \right) \times \left( -\mathbf{C}_r \mathbf{e}_1 + \mathbf{S}_r \mathbf{k} \right) \right]
\]

\[
= -\ell \mathbf{\Omega}_r \mathbf{e}_1 + a \left[ \mathbf{\omega}_r \mathbf{k} + \mathbf{S}_r \mathbf{\omega} \mathbf{e}_1 - \mathbf{\Omega}_r \mathbf{C}_r \mathbf{e}_2 \right]
\]

So,

\[
\ddot{r}_P = \left( a \mathbf{\omega} \mathbf{S}_r - \ell \mathbf{\Omega} \right) \mathbf{e}_1 - \left( a \mathbf{\Omega} \mathbf{C}_r \right) \mathbf{e}_2 + \left( a \mathbf{\omega} \mathbf{C}_r \right) \mathbf{k}
\]

The position vector of \( P \) can be expressed using any convenient set of unit vectors, so the specific form of the position vector is not unique. For example, it could also be written as

\[
\ell_{P/O} = -a \mathbf{C}_r \mathbf{e}_1 + \ell \mathbf{e}_2 + a S_r \mathbf{k}
\]

Differentiating this expression gives

\[
\ddot{r}_P = -\frac{d}{dt} \left( a \mathbf{C}_r \mathbf{e}_1 + \ell \mathbf{e}_2 + a S_r \mathbf{k} \right)
\]

\[
= \left( a \mathbf{S}_r \mathbf{\omega} \right) \mathbf{e}_1 - a \mathbf{C}_r \frac{d}{dt} \left( \mathbf{e}_1 \right) + \frac{d\ell}{dt} \mathbf{e}_2 + \ell \frac{d}{dt} \left( \mathbf{e}_2 \right) + \left( a \mathbf{C}_r \mathbf{\omega} \right) \mathbf{k} + \left( a S_r \mathbf{k} \right) + \left( a S_r \mathbf{\omega} \right) \frac{d}{dt} \left( \mathbf{k} \right)
\]

\[
= \left( a \mathbf{S}_r \mathbf{\omega} \right) \mathbf{e}_1 - a \mathbf{C}_r \left( \mathbf{\Omega}_r \times \mathbf{e}_1 \right) + \ell \left( \mathbf{\Omega}_r \times \mathbf{e}_2 \right) + \left( a \mathbf{C}_r \mathbf{\omega} \right) \mathbf{k}
\]

\[
= \left( a \mathbf{S}_r \mathbf{\omega} \right) \mathbf{e}_1 - a \mathbf{C}_r \mathbf{\Omega}_r \mathbf{e}_2 - \ell \mathbf{\Omega}_r \mathbf{e}_1 + \left( a \mathbf{C}_r \mathbf{\omega} \right) \mathbf{k}
\]

Collecting terms gives the final result.

\[
\ddot{r}_P = \left( a \mathbf{\omega} \mathbf{S}_r - \ell \mathbf{\Omega} \right) \mathbf{e}_1 - \left( a \mathbf{\Omega} \mathbf{C}_r \right) \mathbf{e}_2 + \left( a \mathbf{\omega} \mathbf{C}_r \right) \mathbf{k}
\]
b) The \textit{acceleration} \( \dot{R} \mathbf{a}_p \) is found by differentiating the expression for \( R \mathbf{v}_p \).

\[
\dot{R} \mathbf{a}_p = \frac{d}{dt} ( R \mathbf{v}_p )
\]

\[
= \frac{d}{dt} (a \omega S_\theta - \ell \Omega) \mathbf{e}_1 + (a \omega S_\theta - \ell \Omega) \frac{d}{dt} ( a \omega C_\theta ) \mathbf{e}_2 \\
- \frac{d}{dt} ( a \Omega C_\theta ) \mathbf{e}_2 - ( a \Omega C_\theta ) \frac{d}{dt} ( a \Omega C_\theta ) \mathbf{e}_2 \\
+ \frac{d}{dt} ( a \omega C_\theta ) \mathbf{k} + ( a \omega C_\theta ) \frac{d}{dt} ( a \omega C_\theta ) \mathbf{k}
\]

\[
= \left( a \dot{\omega} S_\theta + a \omega^2 C_\theta - \ell \dot{\Omega} \right) \mathbf{e}_1 + \left( a \omega S_\theta - \ell \Omega \right) \left( \Omega \mathbf{k} \times \mathbf{e}_1 \right) \\
- \left( a \Omega C_\theta - a \omega \Omega S_\theta \right) \mathbf{e}_2 - \left( a \Omega C_\theta - a \omega \Omega S_\theta \right) \left( \Omega \mathbf{k} \times \mathbf{e}_2 \right) + \left( a \dot{\omega} C_\theta - a \omega^2 S_\theta \right) \mathbf{k}
\]

Collecting terms gives the final result.

\[
\dot{R} \mathbf{a}_p = \left[ a \dot{\omega} S_\theta - \ell \dot{\Omega} + a C_\theta ( \omega^2 + \Omega^2 ) \right] \mathbf{e}_1 + \left[ -a \dot{\Omega} C_\theta + 2a \omega \Omega S_\theta - \ell \Omega^2 \right] \mathbf{e}_2 + \left[ a \dot{\omega} C_\theta - a \omega^2 S_\theta \right] \mathbf{k}
\]

\textbf{Derivatives of a Vector in Two Different Reference Frames – The “Derivative Rule”}

Given two reference frames

\( R : (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \) (a rotating frame)
\( S : (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \) (a second rotating frame)

The \textit{derivatives} of any vector \( \mathbf{A} \) in the two reference frames are related as follows

\[
\frac{\dot{R} \mathbf{A}}{dt} = \frac{\delta \mathbf{A}}{dt} + ( R \mathbf{\omega}_S \times \mathbf{A} )
\]

Here \( R \mathbf{\omega}_S \) is the angular velocity of frame \( S \) relative to the frame \( R \).

\textbf{Derivation}

Consider a vector \( \mathbf{A} \) and two reference frames \( R \) and \( S \).

Suppose for convenience \( \mathbf{A} \) is expressed in terms of the unit vectors of frame \( S \). That is,

\[
\mathbf{A} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3
\]

Then, the derivative of \( \mathbf{A} \) in the reference frame \( R \) may be computed as follows
\[
\frac{R \, dA}{dt} = \left( \hat{a}_1 \, \mathbf{e}_1 + \hat{a}_2 \, \mathbf{e}_2 + \hat{a}_3 \, \mathbf{e}_3 \right) + a_1 \left( \frac{R \, d \mathbf{e}_1}{dt} \right) + a_2 \left( \frac{R \, d \mathbf{e}_2}{dt} \right) + a_3 \left( \frac{R \, d \mathbf{e}_3}{dt} \right) \\
= \frac{S \, dA}{dt} + a_1 \left( \frac{R \, \omega_S \times \mathbf{e}_1}{dt} \right) + a_2 \left( \frac{R \, \omega_S \times \mathbf{e}_2}{dt} \right) + a_3 \left( \frac{R \, \omega_S \times \mathbf{e}_3}{dt} \right) \\
= \frac{S \, dA}{dt} + \frac{R \, \omega_S \times \left( a_1 \, \mathbf{e}_1 + a_2 \, \mathbf{e}_2 + a_3 \, \mathbf{e}_3 \right)}{dt} \\
= \frac{S \, dA}{dt} + \left( \frac{R \, \omega_S \times A}{dt} \right)
\]

Note that calculation of \( \frac{R \, dA}{dt} \) requires differentiation of both the scalar functions \( a_i \) \( (i = 1, 2, 3) \) and the unit vectors \( \mathbf{e}_i \) \( (i = 1, 2, 3) \) whereas calculation of \( \frac{S \, dA}{dt} \) requires differentiation of only the scalar functions \( a_i \) \( (i = 1, 2, 3) \) and not the unit vectors \( \mathbf{e}_i \) \( (i = 1, 2, 3) \).

The following example illustrates how to use the derivative rule to calculate the velocity and acceleration of a point within a mechanical system. Its use adds some formality to the differentiation process – it clearly separates differentiation of the scalar and unit vector parts.

Example 2: (Using the Derivative Rule)

Given:
System of Example 1

Find: (express the results using unit vectors fixed in \( F \))

a) \( R \mathbf{v}_P \) …the velocity of point \( P \) in \( R \) using direct differentiation with the derivative rule
b) \( R \mathbf{a}_P \) …the acceleration of point \( P \) in \( R \) using direct differentiation with the derivative rule

Solution:

a) As discussed in the solution to Example 1, the position vector of point \( P \) can be expressed as

\[
\ell_{P/O} = \ell_{Q/O} + \ell_{P/Q} = \ell \, \mathbf{e}_2 + a \, \mathbf{e}_r
\]

The unit vectors \( \mathbf{e}_2 \) and \( \mathbf{e}_r \) are both fixed in the disk \( D \). The derivative rule can be used on this expression to find \( R \mathbf{v}_P \) as follows
\[ R_{p} = \frac{R_{d}}{dt} (r_{p/o}) = \frac{R_{d}}{dt}(\ell \varepsilon_{2} + a \varepsilon_{r}) \]

\[ = \frac{d_{r}}{dt}(\ell \varepsilon_{2} + a \varepsilon_{r}) + \frac{R_{d}}{\omega \times}(\ell \varepsilon_{2} + a \varepsilon_{r}) \]

\[ = (\omega \varepsilon_{2} + \Omega k) \times (\ell \varepsilon_{2} + a \varepsilon_{r}) \]

\[ = (\omega \varepsilon_{2} + \Omega k) \times (\ell \varepsilon_{2} + a (-C_{\theta} \varepsilon_{1} + S_{\theta} k)) \]

\[ = \begin{vmatrix} 
\varepsilon_{1} & \varepsilon_{2} & k \\
0 & \omega & \Omega \\
-aC_{\theta} & \ell & aS_{\theta} 
\end{vmatrix} = (a\omega S_{\theta} - \ell \Omega) \varepsilon_{1} - (a \Omega C_{\theta}) \varepsilon_{2} + (a \omega C_{\theta}) k \]

So, as before,

\[ R_{p} = (a\omega S_{\theta} - \ell \Omega) \varepsilon_{1} - (a \Omega C_{\theta}) \varepsilon_{2} + (a \omega C_{\theta}) k \]

**Alternatively**, the position vector \( r_{p/o} \) can be expressed in terms of unit vectors fixed in \( F \) as

\[ r_{p/o} = -aC_{\theta} \varepsilon_{1} + \ell \varepsilon_{2} + aS_{\theta} k \]

The derivative rule can be applied to this expression along with the chain and product rules to find \( R_{p} \) as follows.

\[ R_{p} = \frac{R_{d}}{dt} (r_{p/o}) = \frac{R_{d}}{dt}(-aC_{\theta} \varepsilon_{1} + \ell \varepsilon_{2} + aS_{\theta} k) \]

\[ = \frac{F_{d}}{dt}(-aC_{\theta} \varepsilon_{1} + \ell \varepsilon_{2} + aS_{\theta} k) + \frac{R_{d}}{\omega \times}(-aC_{\theta} \varepsilon_{1} + \ell \varepsilon_{2} + aS_{\theta} k) \]

\[ = \begin{vmatrix} (a\omega S_{\theta}) \varepsilon_{1} + (a \omega C_{\theta}) k \end{vmatrix} + \begin{vmatrix} (\Omega k) \times (-aC_{\theta} \varepsilon_{1} + \ell \varepsilon_{2} + aS_{\theta} k) \end{vmatrix} \]

\[ = \begin{vmatrix} (a\omega S_{\theta}) \varepsilon_{1} + (a \omega C_{\theta}) k \end{vmatrix} + \begin{vmatrix} (-a \Omega C_{\theta}) \varepsilon_{2} - (\ell \Omega) \varepsilon_{1} \end{vmatrix} \]

\[ = (a\omega S_{\theta} - \ell \Omega) \varepsilon_{1} - (a \Omega C_{\theta}) \varepsilon_{2} + (a \omega C_{\theta}) k \]

So, again,

\[ R_{p} = (a\omega S_{\theta} - \ell \Omega) \varepsilon_{1} - (a \Omega C_{\theta}) \varepsilon_{2} + (a \omega C_{\theta}) k \]

b) The acceleration \( \dot{\omega}_{p} \) can also be calculated using the derivative rule. As the above expressions are both expressed using unit vectors fixed in \( F \), the application of the derivative rule proceeds as follows.

\[ \dot{\omega}_{p} = \frac{R_{d}}{dt} (R_{p}) = \frac{f_{d}}{dt} (R_{p}) + \frac{R_{d}}{\omega \times} \frac{R_{d}}{dt} (R_{p}) \]

where
\[
F \frac{d}{dt} \left( R_{y \rightarrow p} \right) = \frac{d}{dt} \left( (a \omega S_\theta - \ell \Omega) \varepsilon_1 - (a \Omega C_\theta) \varepsilon_2 + (a \omega C_\theta) k \right)
\]
\[
= \left( a \dot{\omega} S_\theta + a \omega^2 C_\theta - \ell \Omega \right) \varepsilon_1 - \left( a \dot{\Omega} C_\theta - a \Omega \omega S_\theta \right) \varepsilon_2
\]
\[
+ \left( a \dot{\omega} C_\theta - a \omega^2 S_\theta \right) k
\]

\[
R_{\Omega F} \times R_{y \rightarrow p} = \left( \Omega k \right) \times \left( (a \omega S_\theta - \ell \Omega) \varepsilon_1 - (a \Omega C_\theta) \varepsilon_2 + (a \omega C_\theta) k \right)
\]
\[
= \left( a \Omega^2 C_\theta \right) \varepsilon_1 + \left( (a \omega S_\theta - \ell \Omega) \Omega \right) \varepsilon_2
\]

Substituting these results into the expression above gives the final result.

\[
R_{\Omega p} = \left[ a \dot{\omega} S_\theta - \ell \dot{\Omega} + a A_\theta \left( \omega^2 + \Omega^2 \right) \right] \varepsilon_1 + \left[ -a \dot{\Omega} C_\theta + 2 a \omega \Omega S_\theta - \ell \Omega^2 \right] \varepsilon_2 + \left[ a \dot{\omega} C_\theta - a \omega^2 S_\theta \right] k
\]

Notes:

1. By this point, it should be clear there is an equivalence between reference frames and rigid bodies. Reference frames are often associated with bodies and move with them. Unit vectors that are fixed in specific bodies are differentiated using the angular velocities of those bodies.
2. The origin of a reference frame may be important, but mostly reference frames are used to indicate directions that are useful for analyzing the motion of a system.
3. Velocity and acceleration vectors (like the angular velocity and angular acceleration vectors) can be expressed using any convenient set of unit vectors. The complexity of the result will depend on the choice of these unit vectors.
4. Using this method, the results for velocity and acceleration are valid for all time and configurations. Thus, the expressions may be more complex than calculations that are valid for only a single instant of time.

Exercises:

2.1 Radial & Transverse Components: The diagram shows two reference frames and a point \( P \) moving along some path. The unit vector set \( R : (i, j, k) \) defines directions in the fixed frame \( R \), and the unit vector set \( E : (\varepsilon_r, \varepsilon_\theta, k) \) defines directions in a rotating frame \( E \). Given that the position vector of \( P \) is \( r_p = r \varepsilon_r \) and using direct differentiation, show the velocity and acceleration of \( P \) can be written as

\[
v_p = \dot{r} \varepsilon_r + r \dot{\theta} \varepsilon_\theta
\]
\[
a_p = \left( \ddot{r} - r \dot{\theta}^2 \right) \varepsilon_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \varepsilon_\theta
\]
2.2 The antenna system shown has two components, the base $B$ and the antenna dish $D$. Base $B$ rotates relative to the ground about the fixed $z$-axis, and dish $D$ rotates relative to $B$ about the rotating $x$-axis. At any instant, the angle between the $y$-axis ($\varepsilon_2$) and the fixed $Y$-axis is $\theta$, and the angle between line $OA$ and the $y$-axis is $\phi$. Given values for $\theta$, $\phi$, and their time derivatives, find $\dot{R}_A$ and $\ddot{R}_A$, the velocity and acceleration of point $A$ in a fixed frame $R$ using direct differentiation.

Answers: 
\[
\dot{R}_A = L \left( \dot{\theta} C_\phi \varepsilon_1 - \dot{\phi} S_\phi \varepsilon_2 + \phi C_\phi k \right)
\]
(results expressed in $B : (\varepsilon_1, \varepsilon_2, k)$)

\[
\ddot{R}_A = L \left( \left( \dot{\theta} C_\phi - 2 \dot{\phi} S_\phi \right) \varepsilon_1 - \left( \ddot{\phi} S_\phi + \phi^2 C_\phi + \theta^2 C_\phi \right) \varepsilon_2 + \left( \dddot{\phi} C_\phi - \phi^2 S_\phi \right) k \right)
\]

2.3 The system shown has three components, a vertical column $C$, a horizontal arm $M$, and a disk $D$. The disk rotates relative to the arm at a rate $\omega_3$ (rad/sec) about the $\varepsilon_3$ direction (normal to $D$), the arm rotates relative to the column at a rate of $\omega_1$ (rad/sec) about the $\varepsilon_1$ direction, and the column rotates relative to the ground at a rate of $\omega_2$ (rad/sec) about the fixed $\varepsilon_2$ direction. The unit vector set $C : (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is fixed in the column, and the unit vector set $M : (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is fixed in arm $M$. Given values for $\omega_1$, $\omega_2$, $\omega_3$, and their time derivatives, find $\dot{R}_A$ and $\ddot{R}_A$, the velocity and acceleration of point $A$ in a fixed frame $R$ using direct differentiation.

Answers: (expressed in $M : (\varepsilon_1, \varepsilon_2, \varepsilon_3)$)

\[
\dot{R}_A = r (\omega_2 S_\phi S_\theta - \omega_3 S_\phi) \varepsilon_1 + (r \omega_3 C_\theta - (L + r C_\theta) \omega_2 S_\phi) \varepsilon_2 + (r \omega_1 S_\theta - (L + r C_\theta) \omega_2 C_\phi) \varepsilon_3
\]

\[
\ddot{R}_A = \left( r \dot{\omega}_2 S_\phi S_\theta - r \omega_2 \dot{S}_\phi S_\theta - r \omega_3 \dot{S}_\phi - (L + r C_\theta) \omega_2^2 + 2 r \omega_1 \omega_2 S_\phi C_\phi + 2 r \omega_2 \omega_3 S_\theta \phi \right) \varepsilon_1 + \\
\left( r \dot{\omega}_3 C_\theta - (L + r C_\theta) \dot{\omega}_2 S_\phi - r \omega_3 \dot{S}_\phi S_\theta - r \omega_2 \dot{S}_\phi S_\theta - r \dot{\omega}_2 S_\phi S_\theta - r \omega_2 S_\phi S_\phi \right) \varepsilon_2 + \\
\left( r \dot{\omega}_1 S_\theta - (L + r C_\theta) \dot{\omega}_2 C_\phi - r \dot{\omega}_2 S_\phi S_\phi - r \omega_2 S_\phi \phi + 2 r \omega_2 \omega_3 S_\theta \phi \right) \varepsilon_3
\]

Hint: Here $\phi$ is the angle between the plane of the disk and the $(\varepsilon_1, \varepsilon_2)$ plane ($\dot{\phi} = \omega_1$). The diagram shows the position where $\phi = 0$. 
References: