An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 2

Velocity and Acceleration Using Direct Differentiation

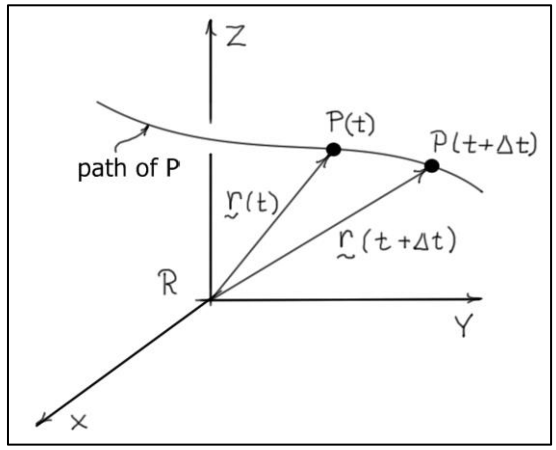
Summary

This unit introduces the concepts of ***velocity*** and ***acceleration*** vectors and shows ***how to calculate*** them using ***direct differentiation***. This technique can be applied to complex mechanical systems; however, at this point the focus will remain on systems in which components are connected by simple revolute (pin) joints. This technique will be generalized in Unit 7 to apply to systems with more complex connecting joints.

Page Count	Examples	Suggested Exercises
8	2	3

Position, Velocity, and Acceleration

The diagram shows the three-dimensional motion of some point P within a mechanical system relative to a reference frame R. Given $\underset{\sim}{r}(t)$ the ***position vector*** of P as a function of time, the ***velocity*** and ***acceleration*** of P relative to R are defined to be



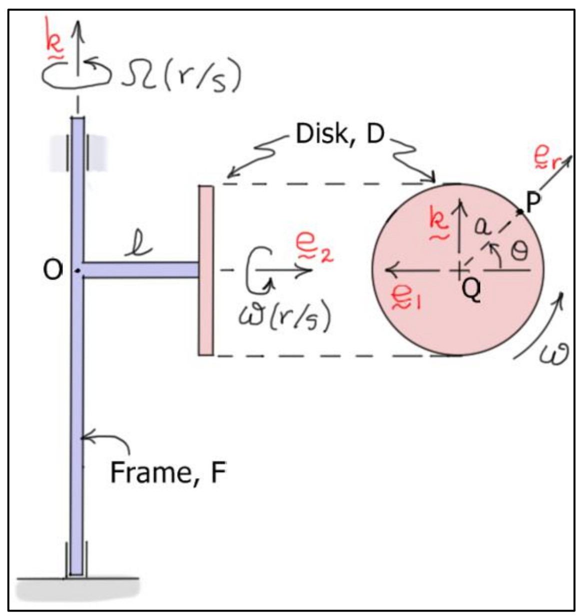
$$\boxed{{}^R\underset{\sim}{v}_P = \frac{{}^R d}{dt}\left(\underset{\sim}{r}(t)\right)} \quad \text{and} \quad \boxed{{}^R\underset{\sim}{a}_P = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_P\right)}$$

The ***velocity*** ${}^R\underset{\sim}{v}_P$ is ***tangent*** to the path of P at all times. The ***acceleration*** ${}^R\underset{\sim}{a}_P$ generally has components ***tangent*** and ***normal*** to the path.

Using these ***fundamental definitions*** along with the concepts for ***differentiating unit vectors*** presented in Unit 1, the ***velocities*** and ***accelerations*** of a points within a mechanical system may be calculated by ***directly differentiating*** their ***position vectors***. The following examples illustrate this process.

Example 1:

The system shown consists of two connected bodies – the frame F and the disk D. Frame F rotates at a rate of Ω (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed to and rotates relative to F at a rate of ω (rad/s) about the horizontal arm of F (annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames:

$R:(\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$ (fixed frame)

$F:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating frame)

Find: (express the results using unit vectors fixed in F)

a) ${}^R\underset{\sim}{v}_P$ …the ***velocity*** of point P in R using ***direct differentiation***

b) ${}^R\underset{\sim}{a}_P$ … the ***acceleration*** of point P in R using ***direct differentiation***

Solution:

a) First, construct a position vector for P that describes its position relative to R in an arbitrary (general) configuration. For example,

$$\boxed{\underset{\sim}{r}_{P/O} = \underset{\sim}{r}_{Q/O} + \underset{\sim}{r}_{P/Q} = \ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r}$$

Here, points O, P, and Q are shown in the diagram, and $\underset{\sim}{e}_r$ is a unit vector pointing from the center of the disk towards P. Differentiating relative to the ground frame R gives

$$\begin{aligned}
{}^{R}\underset{\sim}{v}_P &= \frac{{}^{R}d}{dt}\left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right)\\
&= \ell\,\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_2\right) \quad + a\,\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_r\right)\\
&= \ell\left({}^{R}\underset{\sim}{\omega}_F \times \underset{\sim}{e}_2\right) + a\left[{}^{R}\underset{\sim}{\omega}_D \times \underset{\sim}{e}_r\right]\\
&= \ell\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{e}_2\right) + a\left[\left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right) \times \underset{\sim}{e}_r\right]\\
&= -\ell\,\Omega\,\underset{\sim}{e}_1 + a\left[\left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right) \times \left(-C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{k}\right)\right]\\
&= -\ell\,\Omega\,\underset{\sim}{e}_1 + a\left[\omega C_\theta\,\underset{\sim}{k} + S_\theta\,\omega\,\underset{\sim}{e}_1 - \Omega C_\theta\,\underset{\sim}{e}_2\right]
\end{aligned}$$

So,

$$\boxed{{}^{R}\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}}$$

The position vector of P can be expressed using any convenient set of unit vectors, so the specific form of the position vector is not unique. For example, it could also be written as

$$\boxed{\underset{\sim}{r}_{P/O} = -a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}}$$

Aside: (chain rule)

$$\frac{d}{dt}\left(-a\,C_\theta\right) = -a\left(-S_\theta\,\dot{\theta}\right) = a\,S_\theta\,\omega$$

$$\frac{d}{dt}\left(a\,S_\theta\right) = a\left(C_\theta\,\dot{\theta}\right) = a\,C_\theta\,\omega$$

Differentiating this expression gives

$$\begin{aligned}
{}^{R}\underset{\sim}{v}_P &= \frac{{}^{R}d}{dt}\left(-a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}\right)\\
&= \left(a\,S_\theta\,\omega\right)\underset{\sim}{e}_1 - a\,C_\theta\,\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_1\right) + \underbrace{\frac{d\ell}{dt}}_{\text{zero}}\,\underset{\sim}{e}_2 + \ell\,\frac{{}^{R}d}{dt}\left(\underset{\sim}{e}_2\right) + \left(a\,C_\theta\omega\right)\underset{\sim}{k} + \left(a\,S_\theta\right)\underbrace{\frac{{}^{R}d}{dt}\left(\underset{\sim}{k}\right)}_{\text{zero}}\\
&= \left(a\,S_\theta\,\omega\right)\underset{\sim}{e}_1 - a\,C_\theta\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{e}_1\right) + \ell\left(\Omega\,\underset{\sim}{k} \times \underset{\sim}{e}_2\right) + \left(a\,C_\theta\omega\right)\underset{\sim}{k}\\
&= \left(a\,S_\theta\,\omega\right)\underset{\sim}{e}_1 - a\,C_\theta\,\Omega\,\underset{\sim}{e}_2 - \ell\,\Omega\,\underset{\sim}{e}_1 + \left(a\,C_\theta\omega\right)\underset{\sim}{k}
\end{aligned}$$

Collecting terms gives the final result.

$$\boxed{{}^{R}\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}}$$

b) The ***acceleration*** ${}^R\underset{\sim}{a}_P$ is found by ***differentiating*** the expression for ${}^R\underset{\sim}{v}_P$.

$$
\begin{aligned}
{}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_P\right)\\
&= \frac{d}{dt}\left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 + \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\frac{{}^R d\,\underset{\sim}{e}_1}{dt}\\
&\quad - \frac{d}{dt}\left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 - \left(a\,\Omega\,C_\theta\right)\frac{{}^R d\,\underset{\sim}{e}_2}{dt}\\
&\quad + \frac{d}{dt}\left(a\,\omega\,C_\theta\right)\underset{\sim}{k} + \left(a\,\omega\,C_\theta\right)\underbrace{\frac{{}^R d\,\underset{\sim}{k}}{dt}}_{\text{zero}}\\
&= \left(a\,\dot{\omega}\,S_\theta + a\,\omega^2\,C_\theta - \ell\,\dot{\Omega}\right)\underset{\sim}{e}_1 + \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\overbrace{\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{e}_1\right)}^{\Omega\,\underset{\sim}{e}_2}\\
&\quad - \left(a\,\dot{\Omega}\,C_\theta - a\,\omega\,\Omega\,S_\theta\right)\underset{\sim}{e}_2 - \left(a\,\Omega\,C_\theta\right)\underbrace{\left(\Omega\,\underset{\sim}{k}\times\underset{\sim}{e}_2\right)}_{-\Omega\,\underset{\sim}{e}_1} + \left(a\,\dot{\omega}\,C_\theta - a\,\omega^2\,S_\theta\right)\underset{\sim}{k}
\end{aligned}
$$

Aside: (chain and product rules)

$$
\begin{aligned}
\frac{d}{dt}\left(a\,\omega\,S_\theta\right) &= a\,\dot{\omega}\,S_\theta + a\,\omega\left(\dot{\theta}\,C_\theta\right)\\
&= a\,\dot{\omega}\,S_\theta + a\,\omega^2 C_\theta\\
\frac{d}{dt}\left(a\,\Omega\,C_\theta\right) &= a\,\dot{\Omega}\,C_\theta - a\,\Omega\,\omega\,S_\theta\\
\frac{d}{dt}\left(a\,\omega\,C_\theta\right) &= a\,\dot{\omega}\,C_\theta - a\,\omega\left(\dot{\theta}\,S_\theta\right)\\
&= a\,\dot{\omega}\,C_\theta - a\,\omega^2 S_\theta
\end{aligned}
$$

Collecting terms gives the final result.

$$
\boxed{{}^R\underset{\sim}{a}_P = \left[a\,\dot{\omega}\,S_\theta - \ell\,\dot{\Omega} + a\,C_\theta\left(\omega^2 + \Omega^2\right)\right]\underset{\sim}{e}_1 + \left[-a\,\dot{\Omega}\,C_\theta + 2a\,\omega\,\Omega\,S_\theta - \ell\,\Omega^2\right]\underset{\sim}{e}_2 + \left[a\,\dot{\omega}\,C_\theta - a\,\omega^2 S_\theta\right]\underset{\sim}{k}}
$$

Derivatives of a Vector in Two Different Reference Frames – The "Derivative Rule"

Given two reference frames

$R:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ (a rotating frame)

$S:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ (a second rotating frame)

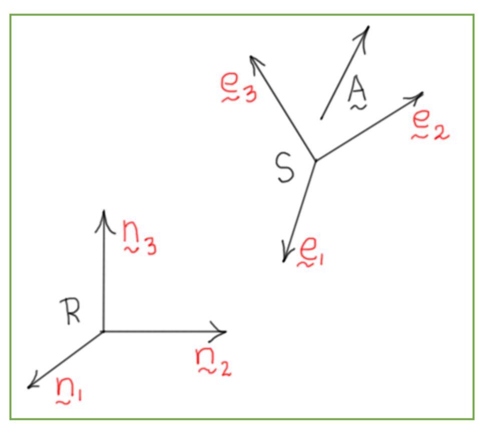
The ***derivatives*** of ***any*** vector $\underset{\sim}{A}$ in the two reference frames are related as follows

$$
\boxed{\frac{{}^R d\underset{\sim}{A}}{dt} = \frac{{}^S d\underset{\sim}{A}}{dt} + \left({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{A}\right)}
$$

Here ${}^R\underset{\sim}{\omega}_S$ is the angular velocity of frame S relative to the frame R.

Derivation

Consider a vector $\underset{\sim}{A}$ and two reference frames R and S. Suppose for convenience $\underset{\sim}{A}$ is expressed in terms of the unit vectors of frame S. That is,



$$
\underset{\sim}{A} = a_1\,\underset{\sim}{e}_1 + a_2\,\underset{\sim}{e}_2 + a_3\,\underset{\sim}{e}_3
$$

Then, the derivative of $\underset{\sim}{A}$ in the reference frame R may be computed as follows

$$\frac{{}^{R}d\underset{\sim}{A}}{dt} = \underbrace{\left(\dot{a}_1\,\underset{\sim}{e}_1 + \dot{a}_2\,\underset{\sim}{e}_2 + \dot{a}_3\,\underset{\sim}{e}_3\right)} + a_1\left(\frac{{}^{R}d\,\underset{\sim}{e}_1}{dt}\right) + a_2\left(\frac{{}^{R}d\,\underset{\sim}{e}_2}{dt}\right) + a_3\left(\frac{{}^{R}d\,\underset{\sim}{e}_3}{dt}\right)$$

$$= \frac{{}^{S}d\underset{\sim}{A}}{dt} + a_1\left({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_1\right) + a_2\left({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_2\right) + a_3\left({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{e}_3\right)$$

$$= \frac{{}^{S}d\underset{\sim}{A}}{dt} + {}^{R}\underset{\sim}{\omega}_S \times \left(a_1\,\underset{\sim}{e}_1 + a_2\,\underset{\sim}{e}_2 + a_3\,\underset{\sim}{e}_3\right)$$

$$= \frac{{}^{S}d\underset{\sim}{A}}{dt} + \left({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{A}\right)$$

Note that calculation of $\frac{{}^{R}d\underset{\sim}{A}}{dt}$ requires differentiation of ***both*** the ***scalar functions*** $a_i\ (i=1,2,3)$ and the ***unit vectors*** $\underset{\sim}{e}_i\ (i=1,2,3)$ whereas calculation of $\frac{{}^{S}d\underset{\sim}{A}}{dt}$ requires differentiation of ***only*** the ***scalar functions*** $a_i\ (i=1,2,3)$ and ***not*** the ***unit vectors*** $\underset{\sim}{e}_i\ (i=1,2,3)$.

The following example illustrates how ***to use*** the ***derivative rule*** to calculate the velocity and acceleration of a point within a mechanical system. Its use adds some ***formality*** to the differentiation process – it clearly ***separates*** differentiation of the ***scalar*** and ***unit vector*** parts.

Example 2: (Using the Derivative Rule)

Given:

System of Example 1

Find: (express the results using unit vectors fixed in *F*)

a) ${}^{R}\underset{\sim}{v}_P$ …the ***velocity*** of point *P* in *R* using ***direct differentiation*** with the ***derivative rule***

b) ${}^{R}\underset{\sim}{a}_P$ …the ***acceleration*** of point *P* in *R* using ***direct differentiation*** with the ***derivative rule***

Solution:

a) As discussed in the solution to Example 1, the ***position vector*** of point *P* can be expressed as

$$\boxed{\underset{\sim}{r}_{P/O} = \underset{\sim}{r}_{Q/O} + \underset{\sim}{r}_{P/Q} = \ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r}$$

The unit vectors $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_r$ are ***both*** fixed in the disk *D*. The derivative rule can be used on this expression to find ${}^{R}\underset{\sim}{v}_P$ as follows

$$
\begin{aligned}
{}^R\underset{\sim}{v}_P &= \frac{{}^R d}{dt}\left(\underset{\sim}{r}_{P/O}\right) = \frac{{}^R d}{dt}\left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right) \\
&= \underbrace{\frac{{}^D d}{dt}\left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right)}_{\text{zero}} + {}^R\underset{\sim}{\omega}_D \times \left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right) \\
&= \left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right) \times \left(\ell\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_r\right) \\
&= \left(\omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right) \times \left(\ell\,\underset{\sim}{e}_2 + a\left(-C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{k}\right)\right) \\
&= \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{k} \\ 0 & \omega & \Omega \\ -a\,C_\theta & \ell & a\,S_\theta \end{vmatrix} = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 + \left(-a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}
\end{aligned}
$$

So, as before,

$$
\boxed{{}^R\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}}
$$

Alternatively, the position vector $\underset{\sim}{r}_{P/O}$ can be expressed in terms of unit vectors fixed in F as

$$
\boxed{\underset{\sim}{r}_{P/O} = -a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}}
$$

The derivative rule can be applied to this expression along with the chain and product rules to find ${}^R\underset{\sim}{v}_P$ as follows.

$$
\begin{aligned}
{}^R\underset{\sim}{v}_P &= \frac{{}^R d}{dt}\left(\underset{\sim}{r}_{P/O}\right) = \frac{{}^R d}{dt}\left(-a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}\right) \\
&= \frac{{}^F d}{dt}\left(-a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}\right) + {}^R\underset{\sim}{\omega}_F \times \left(-a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}\right) \\
&= \left[\left(a\,\omega\,S_\theta\right)\underset{\sim}{e}_1 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}\right] + \left[\left(\Omega\,\underset{\sim}{k}\right) \times \left(-a\,C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a\,S_\theta\,\underset{\sim}{k}\right)\right] \\
&= \left[\left(a\,\omega\,S_\theta\right)\underset{\sim}{e}_1 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}\right] + \left[\left(-a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 - \left(\ell\Omega\right)\underset{\sim}{e}_1\right] \\
&= \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 + \left(-a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}
\end{aligned}
$$

So, again,

$$
\boxed{{}^R\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}}
$$

b) The acceleration ${}^R\underset{\sim}{a}_P$ can also be calculated using the derivative rule. As the above expressions are both expressed using unit vectors fixed in F, the application of the derivative rule proceeds as follows.

$$
\boxed{{}^R\underset{\sim}{a}_P = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_P\right) = \frac{{}^F d}{dt}\left({}^R\underset{\sim}{v}_P\right) + {}^R\underset{\sim}{\omega}_F \times \left({}^R\underset{\sim}{v}_P\right)}
$$

where

$$\frac{{}^F d}{dt}\left({}^R\underset{\sim}{v}_P\right)=\frac{{}^F d}{dt}\Big(\left(a\,\omega\,S_\theta-\ell\,\Omega\right)\underset{\sim}{e}_1-\left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2+\left(a\,\omega\,C_\theta\right)\underset{\sim}{k}\Big)$$
$$=\left(a\,\dot{\omega}\,S_\theta+a\,\omega^2\,C_\theta-\ell\,\dot{\Omega}\right)\underset{\sim}{e}_1-\left(a\,\dot{\Omega}\,C_\theta-a\,\Omega\,\omega\,S_\theta\right)\underset{\sim}{e}_2$$
$$+\left(a\,\dot{\omega}\,C_\theta-a\,\omega^2 S_\theta\right)\underset{\sim}{k}$$
$${}^R\underset{\sim}{\omega}_F\times{}^R\underset{\sim}{v}_P=\left(\Omega\,\underset{\sim}{k}\right)\times\left[\left(a\,\omega\,S_\theta-\ell\,\Omega\right)\underset{\sim}{e}_1-\left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2+\left(a\,\omega\,C_\theta\right)\underset{\sim}{k}\right]$$
$$=\left(a\,\Omega^2 C_\theta\right)\underset{\sim}{e}_1+\left(\left(a\,\omega\,S_\theta-\ell\,\Omega\right)\Omega\right)\underset{\sim}{e}_2$$

Substituting these results into the expression above gives the final result.

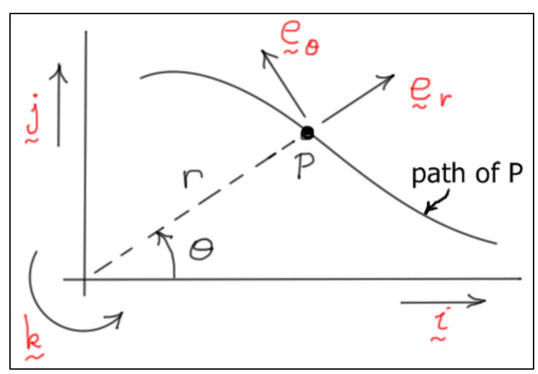
$$\boxed{{}^R\underset{\sim}{a}_P=\left[a\,\dot{\omega}\,S_\theta-\ell\,\dot{\Omega}+a\,C_\theta\left(\omega^2+\Omega^2\right)\right]\underset{\sim}{e}_1+\left[-a\,\dot{\Omega}\,C_\theta+2\,a\,\omega\,\Omega\,S_\theta-\ell\,\Omega^2\right]\underset{\sim}{e}_2+\left[a\,\dot{\omega}\,C_\theta-a\,\omega^2 S_\theta\right]\underset{\sim}{k}}$$

Notes:

1. By this point, it should be clear there is an ***equivalence*** between reference frames and rigid bodies. Reference frames are often ***associated*** with bodies and move with them. ***Unit vectors*** that are fixed in specific bodies are ***differentiated*** using the ***angular velocities*** of those bodies.
2. The ***origin*** of a reference frame may be important, but mostly reference frames are used to ***indicate directions*** that are useful for analyzing the motion of a system.
3. ***Velocity*** and ***acceleration*** vectors (like the angular velocity and angular acceleration vectors) can be expressed using any ***convenient set*** of unit vectors. The ***complexity*** of the result will depend on the choice of these unit vectors.
4. Using this method, the results for velocity and acceleration are valid for ***all time*** and ***configurations***. Thus, the expressions may be more complex than calculations that are valid for only a single instant of time.

Exercises:

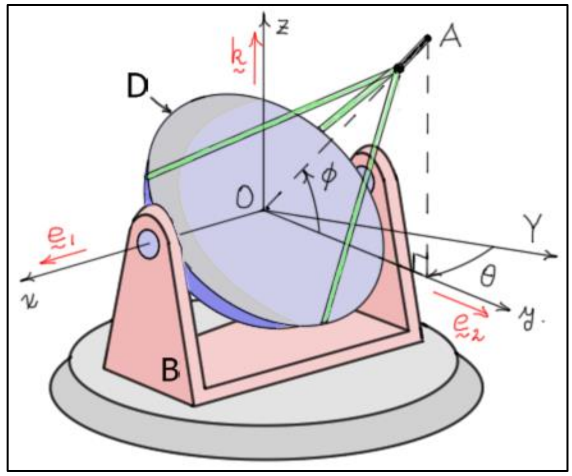
2.1 Radial & Transverse Components: The diagram shows two reference frames and a point *P* moving along some path. The unit vector set $R:(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ defines directions in the fixed frame *R*, and the unit vector set $E:(\underset{\sim}{e}_r,\ \underset{\sim}{e}_\theta,\ \underset{\sim}{k})$ defines directions in a rotating frame *E*. Given that the position vector of *P* is $\underset{\sim}{r}_P=r\,\underset{\sim}{e}_r$ and using ***direct differentiation***, show the velocity and acceleration of *P* can be written as



$$\boxed{\underset{\sim}{v}_P\ =\ \dot{r}\,\underset{\sim}{e}_r\ +\ r\,\dot{\theta}\,\underset{\sim}{e}_\theta}$$

$$\boxed{\underset{\sim}{a}_P\ =\ (\ddot{r}-r\,\dot{\theta}^2)\underset{\sim}{e}_r\ +\ (r\,\ddot{\theta}\ +\ 2\,\dot{r}\,\dot{\theta})\underset{\sim}{e}_\theta}$$

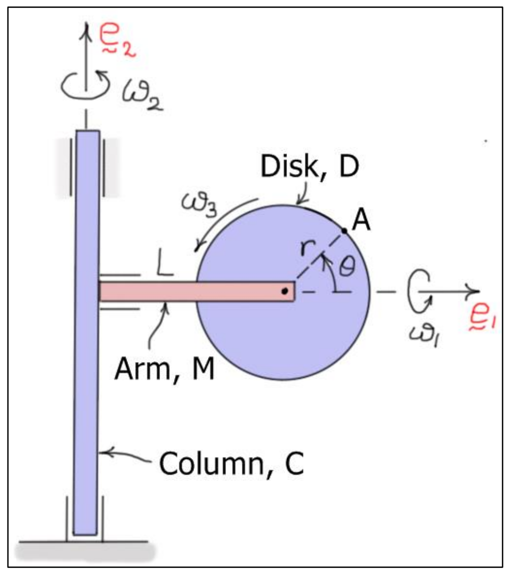
2.2 The antenna system shown has two components, the base B and the antenna dish D. Base B rotates relative to the ground about the fixed z-axis, and dish D rotates relative to B about the rotating x-axis. At any instant, the angle between the y-axis ($\underset{\sim}{e}_2$) and the fixed Y-axis is θ, and the angle between line OA and the y-axis is ϕ. Given values for θ, ϕ, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point A in a fixed frame R using ***direct differentiation***.



Answers: $\boxed{\underset{\sim}{v}_A = L\left(\dot{\theta}C_\phi\,\underset{\sim}{e}_1 - \dot{\phi}S_\phi\,\underset{\sim}{e}_2 + \dot{\phi}C_\phi\,\underset{\sim}{k}\right)}$ (results expressed in $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$)

$$\boxed{\underset{\sim}{a}_A = L\left(\left(\ddot{\theta}C_\phi - 2\dot{\theta}\dot{\phi}S_\phi\right)\underset{\sim}{e}_1 - \left(\ddot{\phi}S_\phi + \dot{\phi}^2C_\phi + \dot{\theta}^2C_\phi\right)\underset{\sim}{e}_2 + \left(\ddot{\phi}C_\phi - \dot{\phi}^2S_\phi\right)\underset{\sim}{k}\right)}$$

2.3 The system shown has three components, a vertical column C, a horizontal arm M, and a disk D. The disk rotates relative to the arm at a rate ω_3 (rad/sec) about the $\underset{\sim}{n}_3$ direction (normal to D), the arm rotates relative to the column at a rate of ω_1 (rad/sec) about the $\underset{\sim}{e}_1$ direction, and the column rotates relative to the ground at a rate of ω_2 (rad/sec) about the fixed $\underset{\sim}{e}_2$ direction. The unit vector set $C:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in the column, and the unit vector set $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ is fixed in arm M. Given values for ω_1, ω_2, ω_3, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point A in a fixed frame R using ***direct differentiation***.



Answers: (expressed in $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$)

$$\boxed{{}^R\underset{\sim}{v}_A = r(\omega_2S_\theta S_\phi - \omega_3S_\theta)\underset{\sim}{e}_1 + (r\omega_3C_\theta - (L+rC_\theta)\omega_2S_\phi)\underset{\sim}{n}_2 + (r\omega_1S_\theta - (L+rC_\theta)\omega_2C_\phi)\underset{\sim}{n}_3}$$

$$\boxed{\begin{aligned}{}^R\underset{\sim}{a}_A = {} & (r\dot{\omega}_2S_\theta S_\phi - r\dot{\omega}_3S_\theta - r\omega_3^2C_\theta - (L+rC_\theta)\omega_2^2 + 2r\omega_1\omega_2S_\theta C_\phi + 2r\omega_2\omega_3S_\phi C_\theta)\underset{\sim}{e}_1 + \\ & (r\dot{\omega}_3C_\theta - (L+rC_\theta)\dot{\omega}_2S_\phi - r\omega_1^2S_\theta - r\omega_2^2S_\theta S_\phi^2 - r\omega_3^2S_\theta + 2r\omega_2\omega_3S_\theta S_\phi)\underset{\sim}{n}_2 + \\ & (r\dot{\omega}_1S_\theta - (L+rC_\theta)\dot{\omega}_2C_\phi - r\omega_2^2S_\theta S_\phi C_\phi + 2r\omega_2\omega_3S_\theta C_\phi + 2r\omega_1\omega_3C_\theta)\underset{\sim}{n}_3\end{aligned}}$$

<u>Hint</u>: Here ϕ is the angle between the plane of the disk and the $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ plane $\left(\dot{\phi} = \omega_1\right)$. The diagram shows the position where $\phi = 0$.

References:

1. T.R. Kane, P.W. Likins, and D.A. Levinson, *Spacecraft Dynamics*, McGraw-Hill, 1983
2. T.R. Kane and D.A. Levinson, *Dynamics: Theory and Application*, McGraw-Hill, 1985
3. R.L. Huston, *Multibody Dynamics*, Butterworth-Heinemann, 1990
4. H. Baruh, *Analytical Dynamics*, McGraw-Hill, 1999
5. H. Josephs and R.L. Huston, *Dynamics of Mechanical Systems*, CRC Press, 2002
6. R.C. Hibbeler, *Engineering Mechanics: Dynamics*, 13th Ed., Pearson Prentice Hall, 2013.