An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume II: Kinetics

Unit 3

Degrees of Freedom, Partial Velocities and Generalized Forces

Summary

This unit defines the concepts of ***degrees of freedom***, ***generalized coordinates***, ***partial velocities***, ***partial angular velocities*** and ***generalized forces***. These concepts form an introduction to methods of treating systems with multiple bodies as ***systems*** rather than one body at a time as with the Newton/Euler equations of motion.

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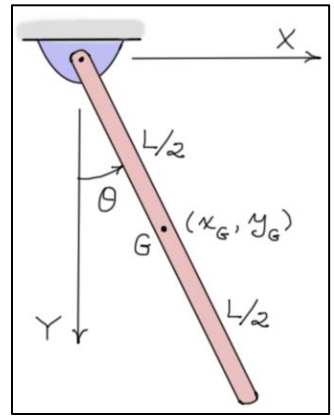
Degrees of Freedom of Mechanical Systems

System Configuration and Generalized Coordinates

Simple Pendulum



The ***configuration*** of a mechanical system is defined as the ***position*** of each of the bodies within the system at a particular instant. In general, both ***translational*** and ***rotational*** coordinates are needed to describe the position of a rigid body. Together the translation and rotation coordinates are called ***generalized coordinates***. For the ***simple pendulum*** shown, x_G and y_G are ***translational coordinates*** that describe the ***location*** of the mass center of the bar, and θ is a ***rotational coordinate*** that describes the ***orientation*** of the bar.

Typically, the ***generalized coordinates*** used to define the ***configuration*** of the mechanical system form a ***dependent*** set. That is, the coordinates are ***not independent of each other***. For example, for the simple pendulum x_G, y_G, and θ are ***not*** independent, because the following two ***independent*** constraint equations can be written

$$\boxed{x_G = \tfrac{1}{2}L\sin(\theta)} \qquad \boxed{y_G = \tfrac{1}{2}L\cos(\theta)}$$

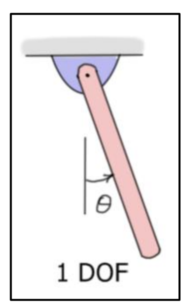
Given the value of one of the coordinates, these equations can be used to compute the values of the other two coordinates, so only ***one*** of the coordinates is needed. ***Any pair*** of these coordinates forms a ***dependent set***.

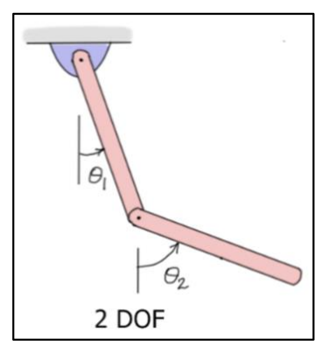
Note:

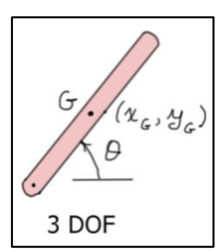
Generalized coordinates ***need not always directly represent translational or rotational variables***, but they will for the purpose of this text.

Generalized Coordinates and Degrees of Freedom

The number of ***degrees of freedom*** (DOF) of a mechanical system is defined as the ***minimum number*** of ***generalized coordinates*** necessary to define the configuration of the system. For a set of generalized coordinates to be ***minimum*** in ***number***, the coordinates must form an ***independent set***. The figures to the right show ***examples*** of one, two, and three degree-of-freedom ***planar systems***.

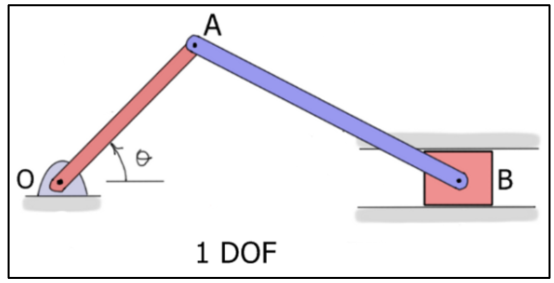






For ***mechanical systems*** that consist of a series of ***interconnected bodies***, it may not be obvious ***how many***

degrees of freedom the system possesses. For these systems, the number of degrees of freedom can be found by first ***calculating*** the number of degrees of freedom the system would possess if the motions of all the bodies were ***unrestricted*** and then ***subtracting*** the ***total number*** of ***constraints*** on the motion. For example, for the two-dimensional ***slider-crank mechanism shown***, the number of degrees of freedom can be calculated in the following ways:

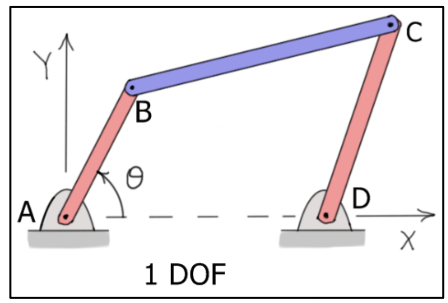


a) Counting 3 bodies: (crank, slider, and connecting bar)

3 bodies @ 3 DOF/body	$= 3\times 3$	= 9 possible DOF
3 pin joints	$= 3\times 2$	= 6 constraints
1 slider joint	$= 1\times 2$	= 2 constraints
Actual DOF	$= 9-6-2$	= 1 DOF

b) Counting 2 bodies: (crank and connecting bar)

2 bodies @ 3 DOF/body	$= 2\times 3$	= 6 possible DOF
2 pin joints	$= 2\times 2$	= 4 constraints
1 pin/slider joint	$= 1\times 1$	= 1 constraint
Actual DOF	$= 6-4-1$	= 1 DOF

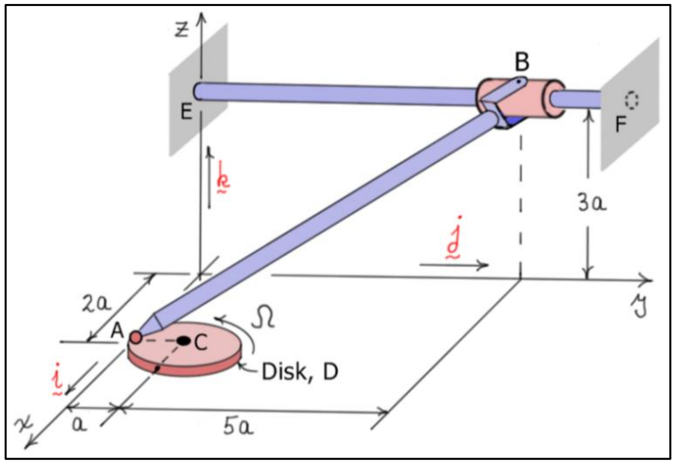


The slider crank mechanism shown is a ***two-dimensional mechanism***. Each pin joint in the mechanism restricts the X and Y motion of that point. Therefore, each ***pin joint*** provides ***two constraint equations***. The ***slider joint*** of part (a) ***restricts*** the ***rotational motion*** and the Y ***translational motion*** of the slider. Hence, it provides ***two constraint equations***. In part (b), the slider is not counted as a body, so the joint at B is considered to ***restrict*** the motion of the ***connecting rod*** AB. The bar is free to translate in the X direction and to rotate at B, but it is ***not free*** to ***translate*** in the Y direction. Hence, the ***pin/slider joint*** provides only ***one constraint equation***.

A similar approach could be used to ***verify*** that the two dimensional ***four-bar mechanism*** shown above also has only ***one degree of freedom***. The system has three moving bars (with 9 possible degrees of freedom) and four pin joints providing eight constraint equations.

This approach can be applied to ***three dimensional systems*** as well. As with two dimensional systems, care must be taken to determine exactly how each joint restricts the motion of the system.

Example 1: Three-Dimensional Slider-Crank Mechanism



The slider-crank mechanism shown has three members – a crank, a slider, and a connecting rod. The crank (disk) is pinned to and rotates about its center C in the xy plane. The slider (collar) at B translates along and rotates about the fixed bar EF. The connecting rod AB is connected to the crank using a ball and socket joint, and it is connected to the slider using a pin joint.

Find:

Calculate the number of degrees of freedom (DOF) of the mechanism.

Solution:

The ***pin joint*** on the ***crank*** at C provides ***five constraints*** by restricting all but the rotational motion of the crank about the z direction relative to the ground. The ***ball and socket joint*** at A provides ***three constraints*** by restricting the x, y and z translations of that point relative to the crank. ***Bar*** EF provides ***four constraints*** by restricting the x and z translations and the x and z rotations of the slider relative to the ground. Finally, the ***pin joint*** at B provides five constraints by restricting all but one rotation of the bar relative to the slider.

Summary:

3 bodies @ 6 DOF/body	$= 3\times 6$	= 18 possible DOF
1 pin joint at C	$= 1\times 5$	= 5 constraints
1 ball and socket joint at A	$= 1\times 3$	= 3 constraints
1 cylindrical joint on slider	$= 1\times 4$	= 4 constraints
1 pin joint at B	$= 1\times 5$	= 5 constraints
Actual DOF	$= 18-5-3-4-5$	= 1 DOF

Partial Velocities

If the ***velocity*** of some point P within a mechanical system can be written in terms of a set of ***generalized coordinates*** q_k $(k=1,\ldots,n)$ and their ***time derivatives*** $\dot{q}_k$ $(k=1,\ldots,n)$, then the ***partial velocities*** of P are defined to be the ***partial derivatives*** of ${}^R\underset{\sim}{v}_P$ with respect to the $\dot{q}_k$ $(k=1,\ldots,n)$.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_P}{\partial \dot{q}_k}} \quad (k=1,\ldots,n)$$

These vectors represent the ***changes*** in the ***velocity*** of P resulting from ***changes*** in each of the $\dot{q}_k$. It can be shown that they also represent the ***changes*** in $\underset{\sim}{r}_P$ the ***position*** of P resulting from ***changes*** in the

$q_k \ (k=1,\ldots,n)$. They are a measure of the ***sensitivity*** of the velocity (or position) of *P* to changes in the $\dot{q}_k$ (or q_k). In this regard, they can also provide a measure of the ***mechanical advantage*** or ***disadvantage*** associated with changes in any of the generalized coordinates. If ***small changes*** in the generalized coordinates produce ***large changes*** in the position of *P*, then the provider has a ***mechanical disadvantage***. If ***large changes*** in the generalized coordinates produce ***small changes*** in the position of *P*, then the provider has a ***mechanical advantage***.

From the perspective of ***multivariate calculus***, if the position vector of *P* is a ***function*** of the ***generalized coordinates*** and ***time***, that is, $\underset{\sim}{r}_P = \underset{\sim}{r}_P(q_k, t)$, then ${}^R\underset{\sim}{v}_P$ the velocity of *P* can be written as follows.

$$\boxed{{}^R\underset{\sim}{v}_P = \frac{{}^R d\underset{\sim}{r}_P}{dt} = \sum_{k=1}^{n}\left(\frac{\partial \underset{\sim}{r}_P}{\partial q_k}\right)\dot{q}_k + \frac{\partial \underset{\sim}{r}_P}{\partial t}}$$

From this result, it also follows that

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{v}_P}{\partial \dot{q}_k} = \frac{\partial \underset{\sim}{r}_P}{\partial q_k}}$$

The term $\dfrac{\partial \underset{\sim}{r}_P}{\partial t}$ accounts for position vectors that depend ***explicitly on time*** (e.g. some specified motion).

Note that since the ***velocity*** is ***linear*** in the $\dot{q}_k \ (k=1,\ldots,n)$, the ***partial velocities*** are ***found*** by ***inspection*** of the velocity vector.

Partial Angular Velocities

Similarly, if the ***angular velocity*** of a body *B* within a mechanical system can be written in terms of a set of ***generalized coordinates*** $q_k \ (k=1,\ldots,n)$ and their ***time derivatives*** $\dot{q}_k \ (k=1,\ldots,n)$, then the ***partial angular velocities*** of *B* are defined to be the ***partial derivatives*** of ${}^R\underset{\sim}{\omega}_B$ with respect to the $\dot{q}_k \ (k=1,\ldots,n)$.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{q}_k} \quad (k=1,\ldots,n)}$$

These vectors represent ***changes*** in the ***angular velocity*** of a body resulting from ***changes*** in the $\dot{q}_k \ (k=1,\ldots,n)$. In general, the angular velocity can be written in terms of the partial angular velocities as

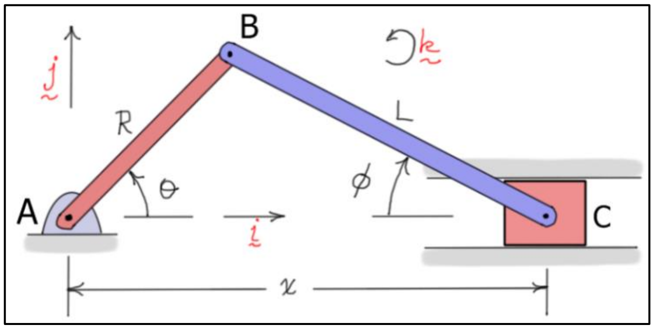
$$\boxed{{}^R\underset{\sim}{\omega}_B = \sum_{k=1}^{n}\left(\frac{\partial\, {}^R\underset{\sim}{\omega}_B}{\partial \dot{q}_k}\right)\dot{q}_k + ({}^R\underset{\sim}{\omega}_B)_t}$$

where $({}^R\underset{\sim}{\omega}_B)_t$ is that part of the angular velocity vector that ***depends explicitly on time***.

Note the above equation ***cannot*** generally be formulated using a ***differentiation process*** as there is no vector whose derivative is the angular velocity. The ***angular velocity*** is formed using the ***summation rule*** for angular velocities, and as with the partial velocities, the ***partial angular velocities*** are ***found by inspection***.

Example 2: Planar, Closed-Loop, Slider Crank Mechanism

System Configuration



The figure shows a simple ***slider crank mechanism*** with no offset. Given the physical dimensions of the links (R, L), the ***configuration*** of the system at any instant of time can be described by ***one*** or ***all*** the ***generalized coordinates*** $\{q_k\} = \{\theta, \phi, x\}$.

As only ***one coordinate*** is required to define the system configuration at any instant, a set of two constraint equations can be written relating the three coordinates. For example, the vector ***loop-closure*** equation $\boxed{\underset{\sim}{r}_{B/A} + \underset{\sim}{r}_{C/B} + \underset{\sim}{r}_{A/C} = \underset{\sim}{0}}$ gives the following two scalar ***constraint equations***.

$$\boxed{\begin{aligned} RS_\theta - LS_\phi &= 0 \\ RC_\theta + LC_\phi - x &= 0 \end{aligned}}$$ (two constraint equations relating the three dependent generalized coordinates)

Here, as previously in this text, the symbols S_θ, S_ϕ, C_θ and C_ϕ have been used to represent the sines and cosines of the angles.

Partial Angular Velocities of the Links

Using the angles shown, the ***angular velocities*** of the crank and connecting bar can be written as $\boxed{{}^R\underset{\sim}{\omega}_{AB} = \dot{\theta}\,\underset{\sim}{k}}$ and $\boxed{{}^R\underset{\sim}{\omega}_{BC} = -\dot{\phi}\,\underset{\sim}{k}}$. From these results, two obvious ***partial angular velocities*** can be defined.

$$\boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{AB}}{\partial \dot{\theta}} = \underset{\sim}{k}} \qquad \boxed{\frac{\partial\, {}^R\underset{\sim}{\omega}_{BC}}{\partial \dot{\phi}} = -\underset{\sim}{k}}$$

A ***second set*** of ***partial angular velocities*** can also be defined by first relating the angular rates $\dot{\theta}$ and $\dot{\phi}$. This can be done by differentiating the first of the two constraint equations with respect to time (using the chain rule).

$$\boxed{R\dot{\theta}C_\theta = L\dot{\phi}C_\phi}$$

Using this result to express one of the angular rates in terms of the other, the following additional partial angular velocities can be defined

$$\boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_{BC}}{\partial\dot{\theta}} = \frac{\partial}{\partial\dot{\theta}}\left(-\dot{\phi}\underset{\sim}{k}\right) = \frac{\partial}{\partial\dot{\theta}}\left[-\left(\frac{RC_\theta}{LC_\phi}\right)\dot{\theta}\underset{\sim}{k}\right] = -\left(\frac{RC_\theta}{LC_\phi}\right)\underset{\sim}{k}} \qquad \boxed{\frac{\partial\,{}^R\underset{\sim}{\omega}_{AB}}{\partial\dot{\phi}} = \frac{\partial}{\partial\dot{\phi}}\left(\dot{\theta}\underset{\sim}{k}\right) = \frac{\partial}{\partial\dot{\phi}}\left[\left(\frac{LC_\phi}{RC_\theta}\right)\dot{\phi}\underset{\sim}{k}\right] = \left(\frac{LC_\phi}{RC_\theta}\right)\underset{\sim}{k}}$$

Note that $\frac{\partial\,{}^R\underset{\sim}{\omega}_{BC}}{\partial\dot{\theta}}$ becomes ***zero*** when the crank angle $\theta = 90\ (\text{deg})$. In this position, the ***connecting rod*** *BC* is ***translating*** and ***not rotating***, so changing the angular velocity of the crank has no effect on the angular velocity of the connecting rod. Conversely, $\frac{\partial\,{}^R\underset{\sim}{\omega}_{AB}}{\partial\dot{\phi}}$ becomes ***undefined*** when the crank angle $\theta = 90\ (\text{deg})$. Near this position, the angular velocity of the crank is ***very sensitive*** to changes in the angular velocity of the connecting rod.

Partial Velocities of the Slider

The ***velocity*** of the slider can be written most simply as $\boxed{{}^R\underset{\sim}{v}_C = \dot{x}\,\underset{\sim}{i}}$. From this result, the following partial velocity can be defined

$$\boxed{\frac{\partial\,{}^R\underset{\sim}{v}_C}{\partial\dot{x}} = \underset{\sim}{i}}$$

Additional partial velocities can be defined by relating $\dot{x}$ to the angular rates $\dot{\theta}$ and $\dot{\phi}$. This can be done by differentiating the second of the constraint equations with respect to time and using the chain rule.

$$\boxed{\dot{x} = -R\,\dot{\theta}\,S_\theta - L\,\dot{\phi}\,S_\phi}$$

Using this result, ${}^R\underset{\sim}{v}_C$ the velocity of the slider can be written as

$$\boxed{{}^R\underset{\sim}{v}_C = \dot{x}\,\underset{\sim}{i} = -\left(R\,\dot{\theta}\,S_\theta + L\,\dot{\phi}\,S_\phi\right)\underset{\sim}{i}}$$

Using this result and the equation above relating the angular rates $\dot{\theta}$ and $\dot{\phi}$, the following additional partial velocities can be defined.

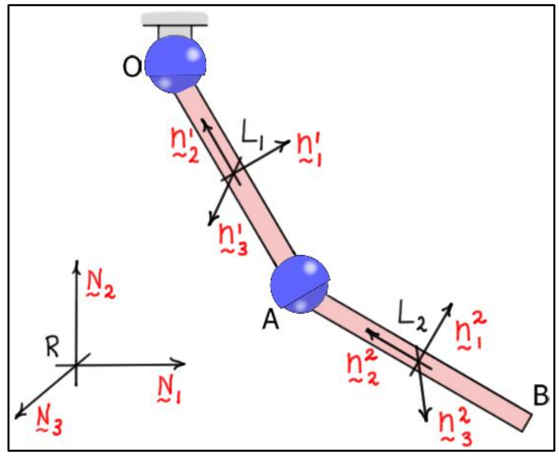
$$ {}^R\underset{\sim}{v}_C = -\dot{\theta}\left[RS_\theta + \not{L}S_\phi\left(\frac{RC_\theta}{\not{L}C_\phi}\right)\right]\underset{\sim}{i} \qquad \Rightarrow \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_C}{\partial\dot{\theta}} = -R\left[S_\theta + C_\theta\,S_\phi/C_\phi\right]\underset{\sim}{i}}$$

$$ {}^R\underset{\sim}{v}_C = -\dot{\phi}\left[\not{R}S_\theta\left(\frac{LC_\phi}{\not{R}C_\theta}\right) + LS_\phi\right]\underset{\sim}{i} \qquad \Rightarrow \boxed{\frac{\partial\,{}^R\underset{\sim}{v}_C}{\partial\dot{\phi}} = -L\left[S_\phi + C_\phi\,S_\theta/C_\theta\right]\underset{\sim}{i}}$$

Note when the crank angle $\theta = 0\ (\text{or }180)\ (\text{deg})$, both of these partial velocities are zero, but when $\theta = \pm 90\ (\text{deg})$, $\frac{\partial\underset{\sim}{v}_C}{\partial\dot{\theta}} = -R\,\underset{\sim}{i}$ and $\frac{\partial\underset{\sim}{v}_C}{\partial\dot{\phi}}$ is undefined. Recall that when $\theta = 0\ (\text{or }180)\ (\text{deg})$, the velocity of the slider is zero and both bars are rotating about their end points, and when $\theta = \pm 90$ (deg) the connecting rod is translating. When the connecting rod is translating, the crank angular velocity has its largest influence on the velocity of the slider.

Example 3: Six DOF Arm

The system shown is a ***six DOF double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***ball and socket*** joints at *O* and *A*. The ground frame is $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ and the link frames are $L_i:\left(\underset{\sim}{n}^i_1, \underset{\sim}{n}^i_2, \underset{\sim}{n}^i_3\right)$ $(i=1,2)$. The ***orientation*** of each link is defined relative to R using a 3-1-3 ***body-fixed*** rotation sequence. The ***lengths*** of the links are ℓ_1 and ℓ_2.



Find:

a) Partial angular velocities of the links associated with the rates of the six orientation angles.

b) Partial velocities of point *B* associated with the rates of the six orientation angles.

Solution:

<u>Previous results</u>:

In Unit 7 of Volume I the ***angular velocities*** of the links and the ***velocity*** of *B* were found to be

$$ {}^R\underset{\sim}{\omega}_{L_i} = \omega_{i1}\,\underset{\sim}{n}^i_1 + \omega_{i2}\,\underset{\sim}{n}^i_2 + \omega_{i3}\,\underset{\sim}{n}^i_3 \quad \text{with} \quad \begin{aligned} \omega_{i1} &= \dot{\theta}_{i1} S_{i2} S_{i3} + \dot{\theta}_{i2} C_{i3} \\ \omega_{i2} &= \dot{\theta}_{i1} S_{i2} C_{i3} - \dot{\theta}_{i2} S_{i3} \\ \omega_{i3} &= \dot{\theta}_{i3} + \dot{\theta}_{i1} C_{i2} \end{aligned} \quad (i=1,2) $$

$$ {}^R\underset{\sim}{v}_B = \left(\ell_1\omega_{13}\,\underset{\sim}{n}^1_1 - \ell_1\omega_{11}\,\underset{\sim}{n}^1_3\right) + \left(\ell_2\omega_{23}\,\underset{\sim}{n}^2_1 - \ell_2\omega_{21}\,\underset{\sim}{n}^2_3\right) $$

a) Using these results, the following ***partial angular velocities*** can be identified.

$$ \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \dot{\theta}_{11}} = S_{12}S_{13}\,\underset{\sim}{n}^1_1 + S_{12}C_{13}\,\underset{\sim}{n}^1_2 + C_{12}\,\underset{\sim}{n}^1_3 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \dot{\theta}_{12}} = C_{13}\,\underset{\sim}{n}^1_1 - S_{13}\,\underset{\sim}{n}^1_2 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_1}}{\partial \dot{\theta}_{13}} = \underset{\sim}{n}^1_3 $$

$$ \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_2}}{\partial \dot{\theta}_{21}} = S_{22}S_{23}\,\underset{\sim}{n}^2_1 + S_{22}C_{23}\,\underset{\sim}{n}^2_2 + C_{22}\,\underset{\sim}{n}^2_3 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_2}}{\partial \dot{\theta}_{22}} = C_{23}\,\underset{\sim}{n}^2_1 - S_{23}\,\underset{\sim}{n}^2_2 \qquad \frac{\partial\, {}^R\underset{\sim}{\omega}_{L_2}}{\partial \dot{\theta}_{23}} = \underset{\sim}{n}^2_3 $$

b) The following partial velocities can also be identified.

$$\frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{11}} = \ell_1 C_{12}\, \underset{\sim}{n}_1^1 - \ell_1 S_{12} S_{13}\, \underset{\sim}{n}_3^1 \qquad \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{12}} = -\ell_1 C_{13}\, \underset{\sim}{n}_3^1 \qquad \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{13}} = \ell_1\, \underset{\sim}{n}_1^1$$

$$\frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{21}} = \ell_2 C_{22}\, \underset{\sim}{n}_1^2 - \ell_2 S_{22} S_{23}\, \underset{\sim}{n}_3^2 \qquad \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{22}} = -\ell_2 C_{23}\, \underset{\sim}{n}_3^2 \qquad \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{23}} = \ell_2\, \underset{\sim}{n}_1^2$$

Recall that matrices $\left[R_1\right]^T$ and $\left[R_2\right]^T$ transform the ***link-based components*** into the ***base frame***. Hence, the results shown for the partial velocities and partial angular velocities could all be easily transformed into the base frame using the transformation matrices.

Generalized Forces

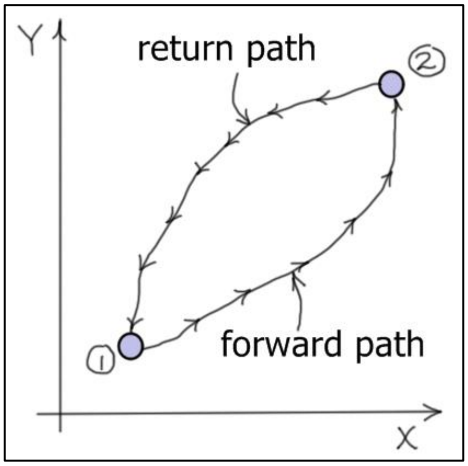
Given a mechanical system whose configuration is defined by a set of generalized coordinates $q_k\ (k=1,...,n)$, the ***generalized forces*** associated with each of the "n" ***generalized coordinates*** as

$$F_{q_k} = \sum_{\substack{forces\\(i)}} \left(\underset{\sim}{F}_i \cdot \frac{\partial\, {}^R\underset{\sim}{v}_i}{\partial \dot{q}_k} \right) + \sum_{\substack{torques\\(j)}} \left(\underset{\sim}{M}_j \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_j}{\partial \dot{q}_k} \right) \qquad (k = 1,\ldots,n)$$

Here the index "i" represents each of the ***forces***, the index "j" represents each of the ***torques***, and the index "k" represents each of the generalized coordinates. The forces $\underset{\sim}{F}_i$ and the torques $\underset{\sim}{M}_j$ that have ***non-zero*** contributions to this sum are said to be ***active***.

Conservative and Nonconservative Forces

Consider a particle that moves from position 1 to position 2 along one path (forward path) and back again to position 1 along a second path (return path) as shown in the diagram. A force acting on the particle is said to be ***conservative*** if the ***net work*** it does over the ***closed path*** is ***zero***. Suppose, for example, that the work done by the force as the particle moves from position 1 to position 2 is ***positive***, then the force does the same amount of work as the particle returns to position 1, except that this work is ***negative***.



In this way, ***conservative forces do not permanently add or remove energy from a system***. When the conservative force is doing ***negative work***, the system is said to be ***gaining potential energy*** that can later be transformed into ***kinetic energy***. It is also true of conservative forces that the work done in moving from one position to another is ***independent*** of the ***path*** of the particle. Examples include ***weight forces*** and ***spring forces*** and ***torques***.

Forces whose ***net work*** around a closed circuit is ***not zero*** are called ***nonconservative forces***. The work done by these forces as a particle moves from one position to another is ***dependent*** on the ***path*** of the particle. Examples include ***friction*** and ***damping*** forces and torques.

Conservative Forces and Potential Energy (V)

The ***generalized force*** associated with ***conservative forces*** and ***torques*** can be written in terms of a ***potential energy function***, V. For weight forces and linear spring forces and torques, the potential energy functions are

$\boxed{V = W\,y = m\,g\,y}$ (y is the ***height*** of the particle ***above*** some ***arbitrary datum***)

$\boxed{V = \frac{1}{2} k e^2}$ (e is the ***elongation*** or ***compression*** of the spring (units of length))

$\boxed{V = \frac{1}{2} k \theta^2}$ (θ is the ***elongation*** or ***compression*** of the spring (radians or degrees))

For systems with ***multiple*** conservative forces, the ***system potential energy*** is $V = \sum_i V_i$.

If conservative forces and torques do work on a system (i.e. if they are ***active***), their contribution to the ***generalized forces*** can be calculated using the ***definition*** given above or they can be calculated using the ***potential energy function*** V .

$$\boxed{F_{q_k} = -\left(\frac{\partial V}{\partial q_k}\right)} \quad (k = 1,\ldots,n)$$

Viscous Damping and Rayleigh's Dissipation Function (R)

One type of nonconservative force or torque is associated with ***viscous damping***. One way of modeling this phenomenon is to ***assume*** that the forces or torques are ***proportional*** to the ***relative velocity*** or ***relative angular velocity*** of the end points of the element.

Force: $\underset{\sim}{F} = -c\,\underset{\sim}{v}_{\text{rel}}$ Torque: $\underset{\sim}{M} = -\left(c\,\dot{\theta}_{\text{rel}}\right)\underset{\sim}{n}$

Here, the direction of the unit vector $\underset{\sim}{n}$ is defined by the right-hand rule. For these types of nonconservative forces and torques, ***Rayleigh's dissipation function*** is

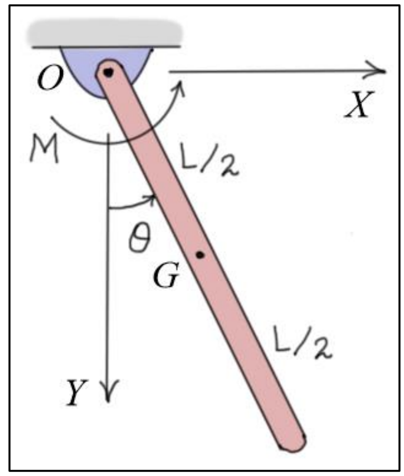
Force: $\boxed{R = \frac{1}{2} c\, \dot{x}_{\text{rel}}^2}$ Torque: $\boxed{R = \frac{1}{2} c\, \dot{\theta}_{\text{rel}}^2}$

Here, the symbols $\dot{x}_{\text{rel}}$ and $\dot{\theta}_{\text{rel}}$ have been used to represent the rates of translational and rotational motion between the ends of the damping element. For systems with ***multiple*** proportional damping elements, the ***system dissipation function*** is $R = \sum_i R_i$.

The contribution of these forces and/or torques to the ***generalized forces*** can be calculated using the ***definition*** given above or they can be calculated using ***Rayleigh's dissipation function***.

$$\boxed{F_{q_k} = -\left(\frac{\partial R}{\partial \dot{q}_k}\right)} \quad (k = 1,\ldots,n)$$

Example 4: Planar, One-Link Pendulum



The one link pendulum shown is acted upon by gravity and by the applied moment M. The Y axis is pointed in the direction of gravity. The single coordinate θ describes the position of the pendulum.

Find:

F_θ, the generalized force associated with the coordinate θ

Solution:

The mechanical system in this case is a single body – the pendulum link. Using the general definition of the generalized force acting on the system associated with the coordinate θ is defined to be

$$F_\theta = \sum_{\substack{forces \\ (i)}}\left(\underset{\sim}{F}_i \cdot \frac{\partial\, {}^R\underset{\sim}{v}_i}{\partial \dot{\theta}}\right) + \sum_{\substack{torques \\ (j)}}\left(\underset{\sim}{M}_j \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_j}{\partial \dot{\theta}}\right) = \left(\underset{\sim}{F}_O \cdot \underbrace{\frac{\partial\, {}^R\underset{\sim}{v}_O}{\partial \dot{\theta}}}_{\text{zero}}\right) + \left(\underset{\sim}{W} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial \dot{\theta}}\right) + \left(\underset{\sim}{M} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}}{\partial \dot{\theta}}\right)$$

The angular velocity of the bar, the velocity of the mass center G can be written as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_B = -\dot{\theta}\,\underset{\sim}{k}} \qquad \boxed{{}^R\underset{\sim}{v}_G = \frac{{}^Rd}{dt}\left(\tfrac{1}{2}LS_\theta\,\underset{\sim}{i} + \tfrac{1}{2}LC_\theta\,\underset{\sim}{j}\right) = \tfrac{1}{2}L\dot{\theta}\left(C_\theta\,\underset{\sim}{i} - S_\theta\,\underset{\sim}{j}\right)}$$

Substituting into the generalized force equation gives

$$\boxed{F_\theta = \left[\underset{\sim}{W} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_G}{\partial \dot{\theta}}\right] + \left[\underset{\sim}{M} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}}{\partial \dot{\theta}}\right] = \left[W\,\underset{\sim}{j} \cdot \tfrac{1}{2}L\left(C_\theta\,\underset{\sim}{i} - S_\theta\,\underset{\sim}{j}\right)\right] + \left[M\,\underset{\sim}{k} \cdot \underset{\sim}{k}\right] = -\tfrac{1}{2}WLS_\theta + M}$$

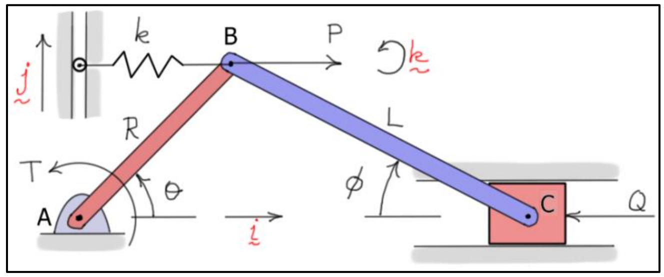
<u>Note</u>:

The contribution of the ***weight force*** could have been calculated using the ***potential energy function*** for gravity. Assuming a ***horizontal datum*** along the X axis passing through O

$$\boxed{\left(F_\theta\right)_{\underset{\sim}{W}} = -\frac{\partial}{\partial \theta}\left(-\tfrac{1}{2}WLC_\theta\right) = -\tfrac{1}{2}WLS_\theta}$$

Example 5: Loaded, Planar, Slider-Crank Mechanism

The figure shows a ***slider crank mechanism*** under the action of an ***external torque*** T acting on link AB, ***external forces*** P and Q acting at B and C, and a ***linear spring*** attached at B. The spring has stiffness k, an unstretched length of ℓ_u, and it always remains horizontal. Neglect weight forces and friction.



Find:

F_θ the generalized force associated with the coordinate θ

Solution:

The ***system*** in this case is the ***entire slider-crank mechanism***. The ***active forces and torques*** acting on the system are the forces P and Q, the torque T, and the linear spring force. The pin forces at A, B and C and the wall force on the slider are ***not active***. Why? The pin at A has zero velocity and zero partial velocity. The pin at B has nonzero velocity and partial velocity, but the internal forces on members AB and BC are equal and opposite, so their net contribution is zero. The same is true for the pin forces at C acting on the BC and the slider. Without friction, the wall force is ***perpendicular*** to the velocity of C, and hence it is not active. So, the generalized force F_θ can be written as follows.

$$
\begin{aligned}
F_\theta &= \sum_{\substack{\text{forces}\\(i)}} \left(\underset{\sim}{F}_i \cdot \frac{\partial\, {}^R\underset{\sim}{v}_i}{\partial \dot{\theta}} \right) + \sum_{\substack{\text{torques}\\(j)}} \left(\underset{\sim}{M}_j \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_j}{\partial \dot{\theta}} \right) = \left(F_\theta\right)_P + \left(F_\theta\right)_Q + \left(F_\theta\right)_T + \left(F_\theta\right)_{\text{spring}} \\
&= \left[P\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}} \right] + \left[-Q\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_C}{\partial \dot{\theta}} \right] + \left[T\, \underset{\sim}{k} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{AB}}{\partial \dot{\theta}} \right] + \left[-F_s\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}} \right]
\end{aligned}
$$

The angular velocity of AB and the velocity of B can be written as

$$
{}^R\underset{\sim}{\omega}_{AB} = \dot{\theta}\, \underset{\sim}{k}
\qquad
{}^R\underset{\sim}{v}_B = {}^R\underset{\sim}{v}_{B/A} = \dot{\theta}\, \underset{\sim}{k} \times R\left(C_\theta\, \underset{\sim}{i} + S_\theta\, \underset{\sim}{j} \right) = R\dot{\theta}\left(-S_\theta\, \underset{\sim}{i} + C_\theta\, \underset{\sim}{j} \right)
$$

The velocity of C was found in Example 2 to be

$$
{}^R\underset{\sim}{v}_C = -R\dot{\theta}\left(S_\theta + C_\theta\, S_\phi / C_\phi \right) \underset{\sim}{i}
$$

Substituting these results into the definition of F_θ gives

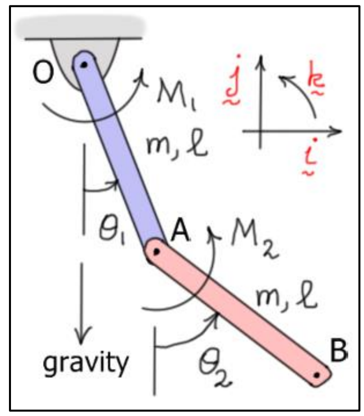
$$
\begin{aligned}
F_\theta &= \left[P\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}} \right] + \left[-Q\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_C}{\partial \dot{\theta}} \right] + \left[T\, \underset{\sim}{k} \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_{AB}}{\partial \dot{\theta}} \right] + \left[-F_s\, \underset{\sim}{i} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}} \right] \\
&= \left[\left(P - F_s \right) \underset{\sim}{i} \cdot R\left(-S_\theta\, \underset{\sim}{i} + C_\theta\, \underset{\sim}{j} \right) \right] + \left[-Q\, \underset{\sim}{i} \cdot -R\left(S_\theta + C_\theta\, S_\phi / C_\phi \right) \underset{\sim}{i} \right] + \left[T\, \underset{\sim}{k} \cdot \underset{\sim}{k} \right]
\end{aligned}
$$

$$\Rightarrow \boxed{F_\theta = \left(F_s - P\right) R S_\theta + QR\left(S_\theta + C_\theta S_\phi / C_\phi\right) + T}$$

The spring force is equal to the product of the spring stiffness and elongation, or $\boxed{F_s = k\left(RC_\theta - \ell_u\right)}$. The contribution of the spring force to F_θ can also be calculated using the ***potential energy function*** of the spring. Specifically,

$$\boxed{\left(F_\theta\right)_{\text{spring}} = -\left(\frac{\partial V_{\text{spring}}}{\partial \theta}\right) = -\frac{\partial}{\partial \theta}\left(\tfrac{1}{2}k\left(RC_\theta - \ell_u\right)^2\right) = -k\left(RC_\theta - \ell_u\right)\left(-RS_\theta\right) = kRS_\theta\left(RC_\theta - \ell_u\right)}$$

Example 6: Planar Arm with Driving Torques



The figure shows a ***two-link arm*** in a vertical plane with ***driving torques*** at the joints. The two ***uniform slender links*** are assumed to be identical with mass m and length ℓ. The system has ***two degrees of freedom*** described by the ***generalized coordinate set*** $\{\theta_1, \theta_2\}$.

Find:

F_{θ_1} and F_{θ_2} the generalized forces associated with θ_1 and θ_2

Solution:

The ***active forces*** in this system are the ***two weight forces*** and the ***two driving torques***. The contribution of the weight forces can be found by using the ***potential energy*** function for gravity. Assuming a ***horizontal datum*** passing through the fixed point *O*, the ***potential energy*** of the system can be written as

$$\boxed{V = V_1 + V_2 = \underbrace{-mg\left(\tfrac{1}{2}\ell C_{\theta_1}\right)}_{V_1} \underbrace{-mg\left(\ell C_{\theta_1} + \tfrac{1}{2}\ell C_{\theta_2}\right)}_{V_2} = -\tfrac{3}{2}mg\ell C_{\theta_1} - \tfrac{1}{2}mg\ell C_{\theta_2}}$$

The contributions of the weight forces to the generalized forces can then be written as

$$\boxed{\left(F_{\theta_1}\right)_{\text{weights}} = -\frac{\partial V}{\partial \theta_1} = -\frac{\partial}{\partial \theta_1}\left(-\tfrac{3}{2}mg\ell C_{\theta_1} - \tfrac{1}{2}mg\ell C_{\theta_2}\right) = -\tfrac{3}{2}mg\ell S_{\theta_1}}$$

$$\boxed{\left(F_{\theta_2}\right)_{\text{weights}} = -\frac{\partial V}{\partial \theta_2} = -\frac{\partial}{\partial \theta_2}\left(-\tfrac{3}{2}mg\ell C_{\theta_1} - \tfrac{1}{2}mg\ell C_{\theta_2}\right) = -\tfrac{1}{2}mg\ell S_{\theta_2}}$$

To calculate the contributions of the driving torques, first note that the torque M_1 is applied to the first link by the ground, and the torque M_2 is applied to the second link by the first link. The torque M_2 applied to the second link is accompanied by a reaction torque $-M_2$ applied to the first link. With this in mind, the ***generalized active forces*** associated with the ***driving torques*** are

$$\left(F_{\theta_1}\right)_{\text{torques}} = \left(M_1\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(-M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1} \right) + \left(M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_1} \right) = \left(M_1\, \underset{\sim}{k} \cdot \underset{\sim}{k} \right) + \left(-M_2\, \underset{\sim}{k} \cdot \underset{\sim}{k} \right) + \left(M_2\, \underset{\sim}{k} \cdot \underset{\sim}{0} \right)$$

$$\Rightarrow \boxed{\left(F_{\theta_1}\right)_{\text{torques}} = M_1 - M_2}$$

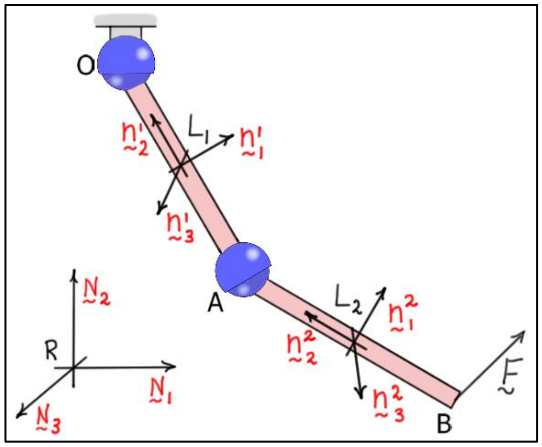
$$\left(F_{\theta_2}\right)_{\text{torques}} = \left(M_1\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(-M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2} \right) + \left(M_2\, \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_2} \right) = \left(M_1\, \underset{\sim}{k} \cdot \underset{\sim}{0} \right) + \left(-M_2\, \underset{\sim}{k} \cdot \underset{\sim}{0} \right) + \left(M_2\, \underset{\sim}{k} \cdot \underset{\sim}{k} \right)$$

$$\Rightarrow \boxed{\left(F_{\theta_2}\right)_{\text{torques}} = M_2}$$

The generalized forces F_{θ_1} and F_{θ_2} are the sums of the above results

$$\boxed{F_{\theta_1} = \left(F_{\theta_1}\right)_{\text{torques}} + \left(F_{\theta_1}\right)_{\text{weights}} = M_1 - M_2 - \tfrac{3}{2} mg\ell S_{\theta_1}} \quad \boxed{F_{\theta_2} = \left(F_{\theta_2}\right)_{\text{torques}} + \left(F_{\theta_2}\right)_{\text{weights}} = M_2 - \tfrac{1}{2} mg\ell S_{\theta_2}}$$

Example 7: Six DOF Arm with End Force



A force $\underset{\sim}{F} = F_1\, \underset{\sim}{N}_1 + F_2\, \underset{\sim}{N}_2 + F_3\, \underset{\sim}{N}_3$ is applied to the end of the six degree of freedom arm described in Example 3.

Find:

$F_{\theta_{1i}}\ (i=1,2,3)$ and $F_{\theta_{2i}}\ (i=1,2,3)$ the generalized forces associated with $\underset{\sim}{F}$ and the six orientation angles

Solution:

In Unit 7 of Volume I, the components of ${}^R\underset{\sim}{v}_B$ the velocity of B in the base frame were found to be

$$\boxed{\begin{Bmatrix} {}^R\underset{\sim}{v}_B \cdot \underset{\sim}{N}_1 \\ {}^R\underset{\sim}{v}_B \cdot \underset{\sim}{N}_2 \\ {}^R\underset{\sim}{v}_B \cdot \underset{\sim}{N}_3 \end{Bmatrix} = \ell_1 \left[R_1\right]^T \begin{Bmatrix} \omega_{13} \\ 0 \\ -\omega_{11} \end{Bmatrix} + \ell_2 \left[R_2\right]^T \begin{Bmatrix} \omega_{23} \\ 0 \\ -\omega_{21} \end{Bmatrix}} \quad \text{with} \quad \boxed{\begin{aligned} \omega_{i1} &= \dot{\theta}_{i1} S_{i2} S_{i3} + \dot{\theta}_{i2} C_{i3} \\ \omega_{i2} &= \dot{\theta}_{i1} S_{i2} C_{i3} - \dot{\theta}_{i2} S_{i3} \\ \omega_{i3} &= \dot{\theta}_{i3} + \dot{\theta}_{i1} C_{i2} \end{aligned}}$$

Using these results, the ***six partial velocities*** of B can be written as follows.

$$\boxed{\begin{Bmatrix} \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{11}} \cdot \underset{\sim}{N}_1 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{11}} \cdot \underset{\sim}{N}_2 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{11}} \cdot \underset{\sim}{N}_3 \end{Bmatrix} = \ell_1 \left[R_1\right]^T \begin{Bmatrix} C_{12} \\ 0 \\ -S_{12} S_{13} \end{Bmatrix}} \quad \boxed{\begin{Bmatrix} \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{12}} \cdot \underset{\sim}{N}_1 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{12}} \cdot \underset{\sim}{N}_2 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{12}} \cdot \underset{\sim}{N}_3 \end{Bmatrix} = \ell_1 \left[R_1\right]^T \begin{Bmatrix} 0 \\ 0 \\ -C_{13} \end{Bmatrix}} \quad \boxed{\begin{Bmatrix} \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{13}} \cdot \underset{\sim}{N}_1 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{13}} \cdot \underset{\sim}{N}_2 \\ \dfrac{\partial\, {}^R\underset{\sim}{v}_B}{\partial \dot{\theta}_{13}} \cdot \underset{\sim}{N}_3 \end{Bmatrix} = \ell_1 \left[R_1\right]^T \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}$$

$$\left\{\begin{matrix}\dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{21}}\cdot\underset{\sim}{N}_1\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{21}}\cdot\underset{\sim}{N}_2\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{21}}\cdot\underset{\sim}{N}_3\end{matrix}\right\}=\ell_2\left[R_2\right]^T\left\{\begin{matrix}C_{22}\\0\\-S_{22}S_{23}\end{matrix}\right\}\qquad \left\{\begin{matrix}\dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{22}}\cdot\underset{\sim}{N}_1\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{22}}\cdot\underset{\sim}{N}_2\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{22}}\cdot\underset{\sim}{N}_3\end{matrix}\right\}=\ell_2\left[R_2\right]^T\left\{\begin{matrix}0\\0\\-C_{23}\end{matrix}\right\}\qquad \left\{\begin{matrix}\dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{23}}\cdot\underset{\sim}{N}_1\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{23}}\cdot\underset{\sim}{N}_2\\[2ex] \dfrac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{23}}\cdot\underset{\sim}{N}_3\end{matrix}\right\}=\ell_2\left[R_2\right]^T\left\{\begin{matrix}1\\0\\0\end{matrix}\right\}$$

Using these results, the ***six generalized forces*** associated with the applied force $\underset{\sim}{F}$ can be written as follows.

$$F_{\theta_{11}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{11}}=\ell_1\left[F_1\quad F_2\quad F_3\right]\left[R_1\right]^T\left\{\begin{matrix}C_{12}\\0\\-S_{12}S_{13}\end{matrix}\right\}$$

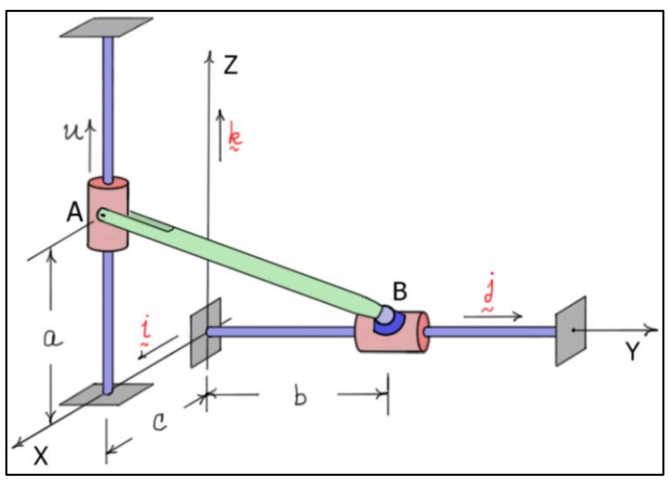
$$F_{\theta_{12}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{12}}=\ell_1\left[F_1\quad F_2\quad F_3\right]\left[R_1\right]^T\left\{\begin{matrix}0\\0\\-C_{13}\end{matrix}\right\}\qquad F_{\theta_{13}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{13}}=\ell_1\left[F_1\quad F_2\quad F_3\right]\left[R_1\right]^T\left\{\begin{matrix}1\\0\\0\end{matrix}\right\}$$

$$F_{\theta_{21}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{21}}=\ell_2\left[F_1\quad F_2\quad F_3\right]\left[R_2\right]^T\left\{\begin{matrix}C_{22}\\0\\-S_{22}S_{23}\end{matrix}\right\}$$

$$F_{\theta_{22}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{22}}=\ell_2\left[F_1\quad F_2\quad F_3\right]\left[R_2\right]^T\left\{\begin{matrix}0\\0\\-C_{23}\end{matrix}\right\}\qquad F_{\theta_{23}}=\underset{\sim}{F}\cdot\frac{\partial\,{}^R\underset{\sim}{v}_B}{\partial\dot{\theta}_{23}}=\ell_2\left[F_1\quad F_2\quad F_3\right]\left[R_2\right]^T\left\{\begin{matrix}1\\0\\0\end{matrix}\right\}$$

Exercises

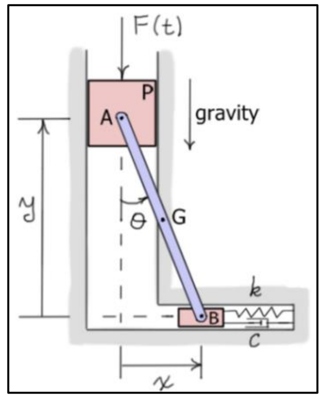
3.1 The system shown consists of ***three bodies*** with ***eighteen*** possible ***degrees of freedom***. The collar at B is connected to the fixed horizontal bar using a ***prismatic joint*** so it can ***translate*** along the bar but not rotate. The collar at A is connected to the fixed vertical bar with a ***cylindrical joint*** so it can translate along and rotate about the bar. Bar AB is connected to the collar at B using a ***ball and socket joint***, and it is connected to the collar at A using a ***pin joint***.



Using the counting procedure discussed above, verify that the system has only ***one degree of freedom***. How many degrees of freedom does the system have if the joint at A is changed to a ball and socket joint? How many degrees of freedom does the system have if the collar at B is connected to the fixed horizontal bar with a cylindrical joint?

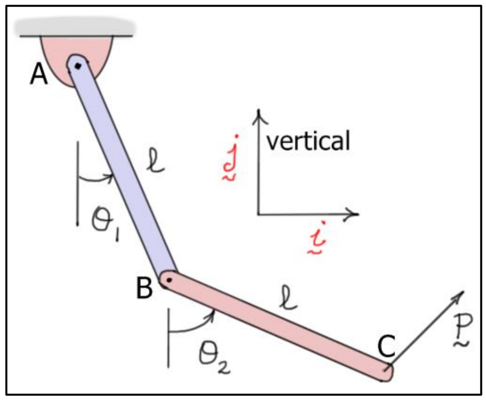
Answers: 1 DOF as is; 3 DOF if ball and socket joint at A; 2 DOF if collar at B can rotate

3.2 The one degree of freedom system shown consists of slender bar AB of mass m and length ℓ and a piston P of mass m_P. The system is driven by the force $F(t)$ and gravity. A spring and damper are attached to the light slider at B. The spring has stiffness k and is unstretched when $x = 0$. Find F_θ the generalized force associated with the coordinate θ. Neglect friction.



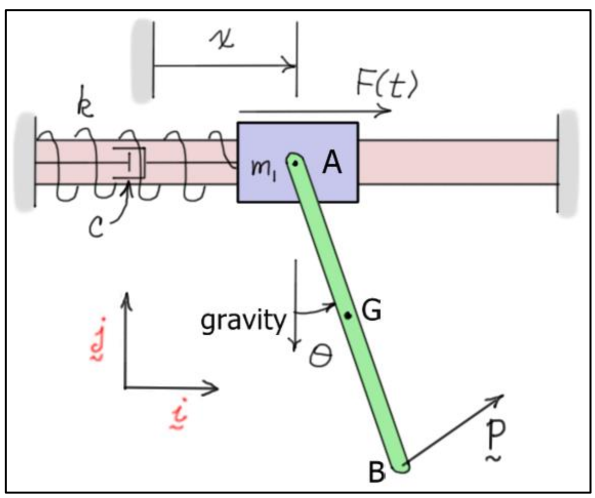
Answer: $F_\theta = F(t)\ell S_\theta + m_P g \ell S_\theta + \frac{1}{2} m g \ell S_\theta - k\ell^2 S_\theta C_\theta - c\,\ell^2 C_\theta^2 \dot{\theta}$

3.3 The system shown is a two degree of freedom pendulum (or arm) that moves in a vertical plane. The external force $\underset{\sim}{P} = P_x \underset{\sim}{i} + P_y \underset{\sim}{j}$ acts on the end of the second link. Find F_{θ_1} and F_{θ_2} the generalized forces associated with the two coordinates θ_1 and θ_2. Include the weight forces and the external force $\underset{\sim}{P}$. The links are identical uniform links with mass m and length ℓ. The mass centers are at the midpoints of the links.



Answers: $F_{\theta_1} = P_x \ell C_{\theta_1} + P_y \ell S_{\theta_1} - \frac{3}{2} m g \ell S_{\theta_1}$ $F_{\theta_2} = P_x \ell C_{\theta_2} + P_y \ell S_{\theta_2} - \frac{1}{2} m g \ell S_{\theta_2}$

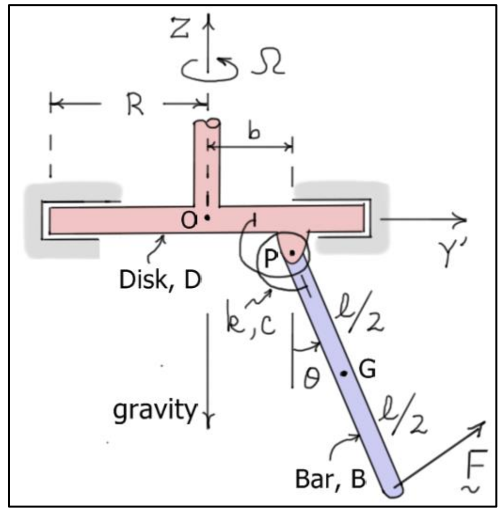
3.4 The system shown consists of a slider of mass m_1 and a uniform slender bar AB of mass m_2 and length ℓ. The slider is connected to the fixed horizontal bar with a prismatic joint – it translates along the bar but does not rotate about it. The slider is attached to the ground by a spring of stiffness k and linear viscous damper with damping coefficient c. The bar is pinned to the slider at A and rotates freely about that point. The system is driven by the forces $\underset{\sim}{F} = F\,\underset{\sim}{i}$ and $\underset{\sim}{P} = P_x \underset{\sim}{i} + P_y \underset{\sim}{j}$. The spring is unstretched at $x = 0$.



Find F_x and F_θ the generalized forces associated with the coordinates x and θ.

Answers: $\boxed{F_x = F + P_x - kx - c\dot{x}}$ $\boxed{F_\theta = P_x \ell C_\theta + P_y \ell S_\theta - \frac{1}{2} m_2 g \ell S_\theta}$

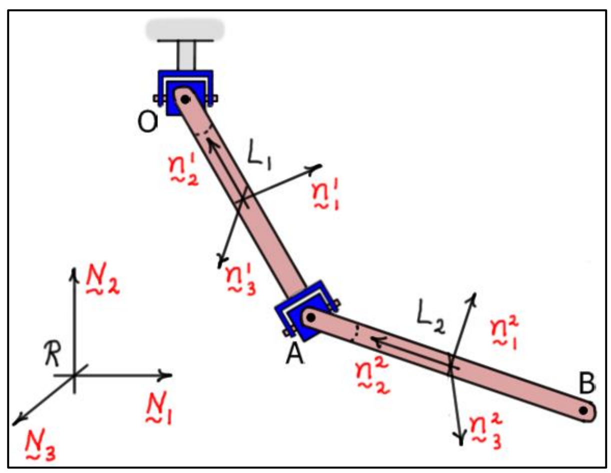
3.5 The two degree of freedom system shown consists of two bodies – disk *D* and slender bar *B*. The disk has radius *R* and mass m_D. The slender bar has length ℓ and mass *m*. The unit vector set $D(\underset{\sim}{i}', \underset{\sim}{j}', \underset{\sim}{k})$ are fixed in the disk along the X', Y' and *Z* axes. The rotation of the disk about the *Z* axis is given by the angle ϕ $(\Omega = \dot{\phi})$, and the rotation of the bar relative to the disk about the X' axis is given by the angle θ. A rotational spring-damper located at pin *P* restricts the motion between the bar and the disk. The spring has stiffness *k* and the linear, viscous damper has damping coefficient *c*. An external force $\underset{\sim}{F} = F_{X'} \underset{\sim}{i}' + F_{Y'} \underset{\sim}{j}' + F_Z \underset{\sim}{k}$ is applied to the end of the bar.



A motor torque M_ϕ is applied by the ground to the disk about the *Z* axis, and a motor torque M_θ is applied by the disk to the bar at *P*. Find F_ϕ and F_θ the generalized forces associated with the coordinates ϕ and θ.

Answers: $\boxed{F_\phi = M_\phi - F_{X'}\left(b + \ell S_\theta\right)}$ $\boxed{F_\theta = M_\theta - \frac{1}{2} m g \ell S_\theta - k\theta - c\dot{\theta} + F_{Y'} \ell C_\theta + F_Z \ell S_\theta}$

3.6 The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***universal joints*** at *O* and *A*, respectively. The ground frame is $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ and the link frames are $L_i:\left(\underset{\sim}{n}^i_1, \underset{\sim}{n}^i_2, \underset{\sim}{n}^i_3\right)$ $(i = 1, 2)$. The ***orientation*** of L_1 is defined ***relative*** to *R* and the orientation of L_2 is defined ***relative*** to L_1 each with a 1-3 ***body-fixed*** rotation sequence.



Link *OA* is oriented relative to the ground frame by first rotating through an angle θ_{11} about the $\underset{\sim}{N}_1$ direction, and then rotating about an angle θ_{12} about the $\underset{\sim}{n}^1_3$ direction. Link *AB* is oriented relative to link *OA* by rotating first through an angle θ_{21} about the $\underset{\sim}{n}^1_1$ direction, and then through an angle θ_{22} about the $\underset{\sim}{n}^2_3$ direction. The lengths of the links are ℓ_1 and ℓ_2 with mass centers are at their midpoints. The system is driven by gravity and by four motor torques, one on each axis of the universal joints. The four motor torques can be written as follows

$\underset{\sim}{M}_{11} = M_{11}\,\underset{\sim}{N}_1$ $\underset{\sim}{M}_{12} = M_{12}\,\underset{\sim}{n}_3^1$ $\underset{\sim}{M}_{21} = M_{21}\,\underset{\sim}{n}_1^1$ $\underset{\sim}{M}_{22} = M_{22}\,\underset{\sim}{n}_3^2$

The two constraint torques transmitted through the joints can be written as

$\underset{\sim}{T}_O = T_O\left(\underset{\sim}{N}_1 \times \underset{\sim}{n}_3^1\right)$ $\underset{\sim}{T}_A = T_A\left(\underset{\sim}{n}_1^1 \times \underset{\sim}{n}_3^2\right)$.

Find $F_{\theta_{11}}$, $F_{\theta_{12}}$, $F_{\theta_{21}}$ and $F_{\theta_{22}}$ the generalized forces associated with the four orientation angles. In the process, show that the contribution of the constraint torques $\underset{\sim}{T}_O$ and $\underset{\sim}{T}_A$ are zero. Assume the $\underset{\sim}{N}_2$ direction is vertical.

Answers:

Recall that matrix $\left[R_1\right]$ transforms components from the base frame (R) to L_1, and the matrix $\left[R_2\right]$ transforms components from L_1 to L_2.

$$\boxed{\begin{aligned}
F_{\theta_{11}} &= M_{11} + \tfrac{1}{2}\ell_1\begin{bmatrix}0 & -W_1 & 0\end{bmatrix}\left[R_1\right]^T\begin{Bmatrix}0\\0\\-C_{12}\end{Bmatrix}\\
&\quad+\ell_1\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\begin{Bmatrix}0\\0\\-C_{12}\end{Bmatrix}+\tfrac{1}{2}\ell_2\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\left[R_2\right]^T\begin{Bmatrix}S_{12}S_{21}\\0\\S_{12}C_{21}S_{22}-C_{12}C_{22}\end{Bmatrix}\\
&= M_{11} - \tfrac{1}{2}\ell_1 W_1 S_{11}C_{12} - \left[\ell_1 W_2 S_{11}C_{12} - \tfrac{1}{2}\ell_2 W_2\left(S_{22}S_{11}S_{12} - C_{21}C_{22}S_{11}C_{12} - S_{21}C_{22}C_{11}\right)\right]
\end{aligned}}$$

$$\boxed{\begin{aligned}
F_{\theta_{12}} &= M_{12} + \tfrac{1}{2}\ell_1\begin{bmatrix}0 & -W_1 & 0\end{bmatrix}\left[R_1\right]^T\begin{Bmatrix}1\\0\\0\end{Bmatrix}\\
&\quad+\ell_1\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\begin{Bmatrix}1\\0\\0\end{Bmatrix}+\tfrac{1}{2}\ell_2\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\left[R_2\right]^T\begin{Bmatrix}C_{21}\\0\\-S_{21}S_{22}\end{Bmatrix}\\
&= M_{12} - \tfrac{1}{2}\ell_1 W_1 C_{11}S_{12} - \left[\ell_1 W_2 C_{11}S_{12} - \tfrac{1}{2}\ell_2 W_2\left(-S_{22}C_{11}C_{12} - C_{21}C_{22}C_{11}S_{12}\right)\right]
\end{aligned}}$$

$$\boxed{F_{\theta_{21}} = M_{21} + \tfrac{1}{2}\ell_2\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\left[R_2\right]^T\begin{Bmatrix}0\\0\\-C_{22}\end{Bmatrix} = M_{21} - \tfrac{1}{2}\ell_2 W_2\left(S_{21}C_{22}C_{11}C_{12} + C_{21}C_{22}S_{11}\right)}$$

$$\boxed{\begin{aligned}
F_{\theta_{22}} &= M_{22} + \tfrac{1}{2}\ell_2\begin{bmatrix}0 & -W_2 & 0\end{bmatrix}\left[R_1\right]^T\left[R_2\right]^T\begin{Bmatrix}1\\0\\0\end{Bmatrix}\\
&= M_{22} - \tfrac{1}{2}\ell_2 W_2\left(C_{22}C_{11}S_{12} + C_{21}S_{22}C_{11}C_{12} - S_{21}S_{22}S_{11}\right)
\end{aligned}}$$

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