An Introduction to
Three-Dimensional, Rigid Body Dynamics

James W. Kamman, PhD

Volume I: Kinematics

Unit 3

Relative Kinematics of Two Points Fixed on a Rigid Body

Summary

This unit continues the development of the concepts of ***velocity*** and ***acceleration*** vectors and shows ***how to calculate*** them using ***formulae*** that relate the motion of ***two points fixed*** on a rigid body. As before, this technique can be applied to complex mechanical systems; however, at this point the focus will remain on systems in which components are connected by simple revolute (pin) joints. This technique will be ***generalized*** in Unit 7 to apply to systems with more ***complex connecting joints***.

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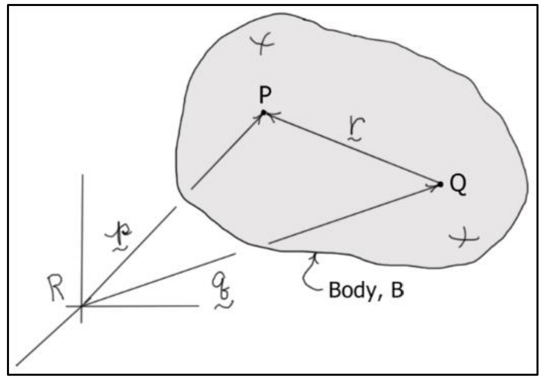
Relative Kinematics of Two Points Fixed on a Rigid Body

General Concept

Consider the three-dimensional motion of a rigid body B as shown in the diagram at the right. The points P and Q represent two points that are fixed in the body. The velocities and accelerations of P and Q in the reference frame R are related as follows



$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{v}_{P/Q} = {}^R\underset{\sim}{v}_Q + ({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r})}$$

and

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_Q + {}^R\underset{\sim}{a}_{P/Q} = {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)}$$

These equations are easily verified using the "***derivative rule***" discussed in Unit 2.

Derivation

The ***position vector*** of point P relative to the reference frame R can be written as $\underset{\sim}{p} = \underset{\sim}{q} + \underset{\sim}{r}$. ***Differentiating*** this equation and using the "*derivative rule*" gives

$$\left.\begin{aligned}
{}^R\underset{\sim}{v}_P &= \frac{{}^R d\,\underset{\sim}{p}}{dt} \\
&= \frac{{}^R d\,\underset{\sim}{q}}{dt} + \frac{{}^R d\,\underset{\sim}{r}}{dt} \\
&= {}^R\underset{\sim}{v}_Q + \underbrace{\left(\frac{{}^B d\,\underset{\sim}{r}}{dt}\right)}_{\text{zero}} + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \\
&= {}^R\underset{\sim}{v}_Q + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)
\end{aligned}\right\} \Rightarrow \boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P/Q} = \underset{\sim}{v}_Q + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right)}$$

where

$$\boxed{\frac{{}^R d\,\underset{\sim}{r}}{dt} = {}^R\underset{\sim}{v}_{P/Q} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}}$$

(velocity of P ***relative*** to Q in R, ${}^R\underset{\sim}{v}_{P/Q}$)

Differentiating the velocity equation and again using the "***derivative rule***" gives

$$\begin{aligned}
{}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_P) = \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_Q) + \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \\
&= {}^R\underset{\sim}{a}_Q + \left(\frac{{}^R d}{dt}({}^R\underset{\sim}{\omega}_B) \times \underset{\sim}{r}\right) + \left({}^R\underset{\sim}{\omega}_B \times \frac{{}^R d\underset{\sim}{r}}{dt}\right) \\
&= {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)
\end{aligned}$$

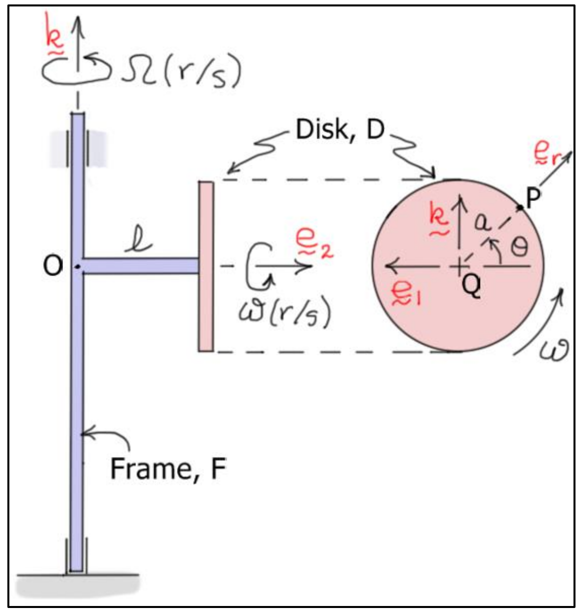
or

$$\boxed{{}^{R}\underset{\sim}{a}_{P} = {}^{R}\underset{\sim}{a}_{Q} + {}^{R}\underset{\sim}{a}_{P/Q} = {}^{R}\underset{\sim}{a}_{Q} + \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)}$$

Here ${}^{R}\underset{\sim}{a}_{P/Q}$ is the acceleration of P with respect to Q in R, and by inspection of the above equation, it is defined to be

$$\boxed{{}^{R}\underset{\sim}{a}_{P/Q} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P/Q}\right) = \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)}$$

Example:



The system shown consists of two connected bodies – the frame F and the disk D. Frame F rotates at a rate of Ω (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed to and rotates relative to F at a rate of ω (rad/s) about the horizontal arm of F (annotated by the rotating unit vector $\underset{\sim}{e}_2$).

Reference frames:

$R: (\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ (fixed frame)

$F: (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating frame)

Find: (express the results using unit vectors fixed in F)

a) ${}^{R}\underset{\sim}{v}_{P}$ … the ***velocity*** of point P in R using the ***two-point formula***

b) ${}^{R}\underset{\sim}{a}_{P}$ … the ***acceleration*** of point P in R using the ***two-point formula***

Solution:

a) Using the ***two-point formula*** for ***velocity***, the velocity of P can be written as

$$\boxed{{}^{R}\underset{\sim}{v}_{P} = {}^{R}\underset{\sim}{v}_{Q} + {}^{R}\underset{\sim}{v}_{P/Q}}$$

where

$$\boxed{{}^{R}\underset{\sim}{v}_{Q} = -\ell\,\Omega\,\underset{\sim}{e}_1}$$ (Q has ***circular motion*** around O)

Using the result from Unit 1 for the angular velocity of D $\left({}^{R}\underset{\sim}{\omega}_{D} = \omega\,\underset{\sim}{e}_2 + \Omega\,\underset{\sim}{k}\right)$, the velocity of P relative to Q may be written as

$$\boxed{\begin{aligned}{}^{R}\underset{\sim}{v}_{P/Q} = {}^{R}\underset{\sim}{\omega}_{D} \times \underset{\sim}{r}_{P/Q} &= \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{k} \\ 0 & \omega & \Omega \\ -a\,C_\theta & 0 & a\,S_\theta \end{vmatrix} \\ &= \left(a\,\omega\,S_\theta\right)\underset{\sim}{e}_1 + \left(-a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k}\end{aligned}}$$

Adding these two results gives the same result as found in Unit 2.

$$ {}^R\underset{\sim}{v}_P = \left(a\,\omega\,S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\,C_\theta\right)\underset{\sim}{k} $$

b) Using the two-point formula for ***acceleration***, the acceleration of *P* can be written as

$$ {}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_D \times \underset{\sim}{r}_{P/Q}\right) + \left({}^R\underset{\sim}{\omega}_D \times {}^R\underset{\sim}{v}_{P/Q}\right) $$

where

$$ {}^R\underset{\sim}{a}_Q = -\ell\,\dot{\Omega}\,\underset{\sim}{e}_1 - \ell\,\Omega^2\,\underset{\sim}{e}_2 $$ (Q has ***circular motion*** around O)

Using the results from Unit 1 for the ***angular velocity*** of *D* in *R* and the ***angular acceleration*** of *D* in *R* $\left({}^R\underset{\sim}{\alpha}_D = -\omega\,\Omega\,\underset{\sim}{e}_1 + \dot{\omega}\,\underset{\sim}{e}_2 + \dot{\Omega}\,\underset{\sim}{k}\right)$, the acceleration of *P* with respect to *Q* may be calculated using the following terms

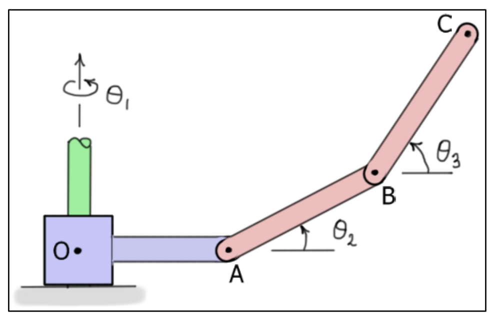
$$ {}^R\underset{\sim}{\alpha}_D \times \underset{\sim}{r}_{P/Q} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{k} \\ -\omega\,\Omega & \dot{\omega} & \dot{\Omega} \\ -a\,C_\theta & 0 & a\,S_\theta \end{vmatrix} = \left(a\,\dot{\omega}\,S_\theta\right)\underset{\sim}{e}_1 + \left(a\,\omega\,\Omega\,S_\theta - a\,\dot{\Omega}\,C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\dot{\omega}\,C_\theta\right)\underset{\sim}{k} $$

$$ {}^R\underset{\sim}{\omega}_D \times {}^R\underset{\sim}{v}_{P/Q} = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{k} \\ 0 & \omega & \Omega \\ a\,\omega\,S_\theta & -a\,\Omega\,C_\theta & a\,\omega\,C_\theta \end{vmatrix} = \left(a\,\omega^2\,C_\theta + a\,\Omega^2\,C_\theta\right)\underset{\sim}{e}_1 + \left(a\,\omega\,\Omega\,S_\theta\right)\underset{\sim}{e}_2 + \left(-a\,\omega^2\,S_\theta\right)\underset{\sim}{k} $$

Substituting these results into the boxed equation gives the same result as found in Unit 2.

$$ {}^R\underset{\sim}{a}_P = \left[a\,\dot{\omega}\,S_\theta - \ell\,\dot{\Omega} + a\,C_\theta\left(\omega^2 + \Omega^2\right)\right]\underset{\sim}{e}_1 + \left[-a\,\dot{\Omega}\,C_\theta + 2\,a\,\omega\,\Omega\,S_\theta - \ell\,\Omega^2\right]\underset{\sim}{e}_2 + \left[a\,\dot{\omega}\,C_\theta - a\,\omega^2 S_\theta\right]\underset{\sim}{k} $$

Notes



1. When bodies are adjoined such that a ***single point*** can be identified that is common to the two bodies, this point can be used to "step" from one body to the next. In this manner, the above formulae may be applied ***recursively*** to calculate motions of remote points within a mechanical system. For example, the velocity of point *C* in the system shown can be written as

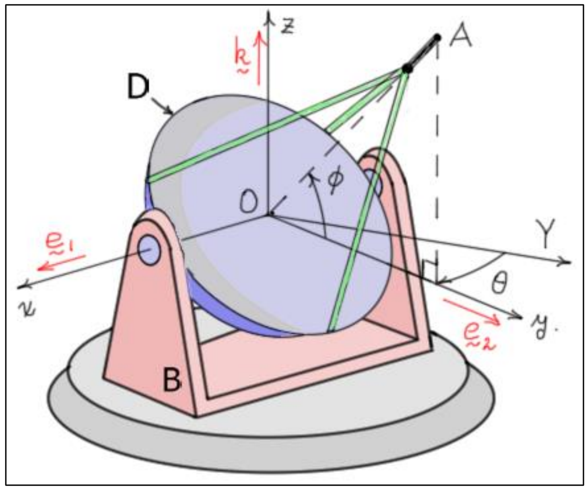
$$ {}^R\underset{\sim}{v}_C = {}^R\underset{\sim}{v}_B + {}^R\underset{\sim}{v}_{C/B} = {}^R\underset{\sim}{v}_A + {}^R\underset{\sim}{v}_{B/A} + {}^R\underset{\sim}{v}_{C/B} = \underbrace{{}^R\underset{\sim}{v}_O}_{\text{zero}} + {}^R\underset{\sim}{v}_{A/O} + {}^R\underset{\sim}{v}_{B/A} + {}^R\underset{\sim}{v}_{C/B} $$

   A similar approach may be taken for the acceleration of point *C*.

2. Using this approach, calculation of velocities and accelerations ***does not require differentiation***. It requires only ***vector multiplication*** and ***addition***. Results may be calculated that are valid for ***all time*** or only a ***specific instant*** of time. Recall that when using direct differentiation, the method requires the results be valid for all time.
3. In the sequel, the formulae above will be referred as the "***two-point***" formulae.

Exercises:

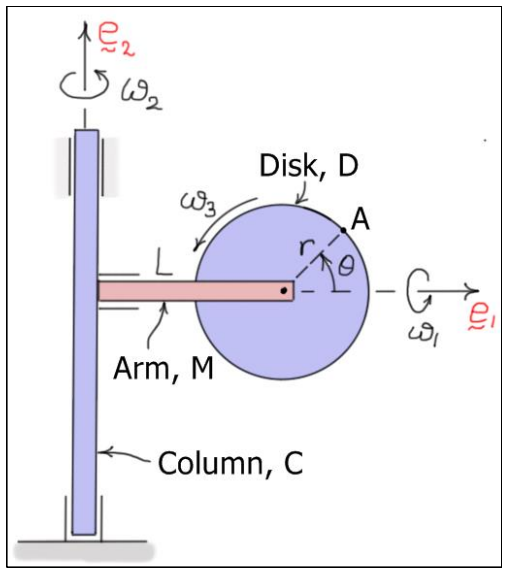
3.1 The antenna system shown has two components, the base B and the antenna dish D. Base B rotates relative to the ground about the fixed z-axis, and dish D rotates relative to B about the rotating x-axis. At any instant, the angle between the y-axis ($\underset{\sim}{e}_2$) and the fixed Y-axis is θ, and the angle between line OA and the y-axis is ϕ. Given values for θ, ϕ, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point A in a fixed frame R using the ***two-point formulae***.



Answers: $\boxed{\underset{\sim}{v}_A = L\left(\dot{\theta}C_\phi \underset{\sim}{e}_1 - \dot{\phi}S_\phi \underset{\sim}{e}_2 + \dot{\phi}C_\phi \underset{\sim}{k}\right)}$ (results expressed in $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$)

$$\boxed{\underset{\sim}{a}_A = L\left(\left(\ddot{\theta}C_\phi - 2\dot{\theta}\dot{\phi}S_\phi\right)\underset{\sim}{e}_1 - \left(\ddot{\phi}S_\phi + \dot{\phi}^2C_\phi + \dot{\theta}^2C_\phi\right)\underset{\sim}{e}_2 + \left(\ddot{\phi}C_\phi - \dot{\phi}^2S_\phi\right)\underset{\sim}{k}\right)}$$

3.2 The system shown has three components, a vertical column C, a horizontal arm M, and a disk D. The disk rotates relative to the arm at a rate ω_3 (rad/sec) about the $\underset{\sim}{n}_3$ direction (normal to D), the arm rotates relative to the column at a rate of ω_1 (rad/sec) about the $\underset{\sim}{e}_1$ direction, and the column rotates relative to the ground at a rate of ω_2 (rad/sec) about the fixed $\underset{\sim}{e}_2$ direction. The unit vector set $C:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in the column, and the unit vector set $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ is fixed in arm M. Given values for ω_1, ω_2, ω_3, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point A in a fixed frame R using the ***two-point formulae***.



Answers: (expressed in $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$)

$$\boxed{{}^R\underset{\sim}{v}_A = r(\omega_2 S_\theta S_\phi - \omega_3 S_\theta)\underset{\sim}{e}_1 + (r\omega_3 C_\theta - (L + rC_\theta)\omega_2 S_\phi)\underset{\sim}{n}_2 + (r\omega_1 S_\theta - (L + rC_\theta)\omega_2 C_\phi)\underset{\sim}{n}_3}$$

$$\boxed{\begin{aligned}{}^R\underset{\sim}{a}_A = &(r\dot{\omega}_2 S_\theta S_\phi - r\dot{\omega}_3 S_\theta - r\omega_3^2 C_\theta - (L + rC_\theta)\omega_2^2 + 2r\omega_1\omega_2 S_\theta C_\phi + 2r\omega_2\omega_3 S_\phi C_\theta)\underset{\sim}{e}_1 + \\ &(r\dot{\omega}_3 C_\theta - (L + rC_\theta)\dot{\omega}_2 S_\phi - r\omega_1^2 S_\theta - r\omega_2^2 S_\theta S_\phi^2 - r\omega_3^2 S_\theta + 2r\omega_2\omega_3 S_\theta S_\phi)\underset{\sim}{n}_2 + \\ &(r\dot{\omega}_1 S_\theta - (L + rC_\theta)\dot{\omega}_2 C_\phi - r\omega_2^2 S_\theta S_\phi C_\phi + 2r\omega_2\omega_3 S_\theta C_\phi + 2r\omega_1\omega_3 C_\theta)\underset{\sim}{n}_3\end{aligned}}$$

Hint: Here ϕ is the angle between the plane of the disk and the $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ plane $\left(\dot{\phi} = \omega_1\right)$. The diagram shows the position where $\phi = 0$.

References:

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