An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 4

Kinematics of a Point Moving on a Rigid Body

Summary

This unit continues the development of the concepts of ***velocity*** and ***acceleration*** vectors and shows ***how to calculate*** them using formulae for points ***moving on non-stationary rigid bodies***. As before, this technique can be applied to complex mechanical systems; however, at this point the focus will remain on systems in which components are connected by simple revolute (pin) joints. This technique will be ***generalized*** in Unit 7 to apply to systems with more ***complex connecting joints***.

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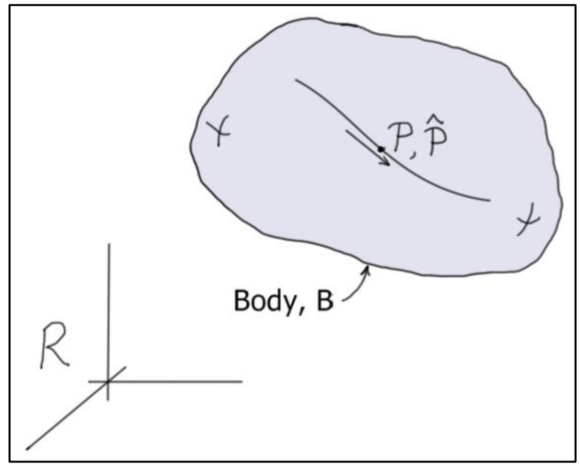
Kinematics of a Point Moving on a Rigid Body

General Concept



The kinematic analysis is now extended to include systems where interconnected bodies may ***rotate*** and ***translate*** relative to each other. In this case, the kinematics of points that are moving on (or relative to) rotating bodies is needed. To analyze this motion, consider the figure shown at the right. Here,

R : is a fixed reference frame,

B : is a rigid body,

P : is a point ***moving*** on B, and

$\hat{P}$: is a point ***fixed*** on B that coincides with P at this instant of time

The velocity and acceleration of P may be written as

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P}$$

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)}$$

where each of the terms are defined as follows

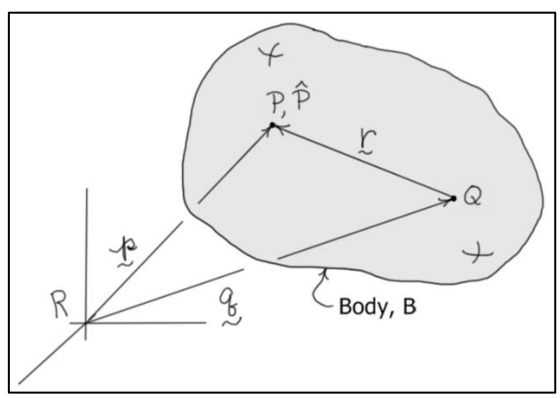
${}^B\underset{\sim}{v}_P,\ {}^B\underset{\sim}{a}_P$: the velocity and acceleration of P on B, assuming B is fixed

${}^R\underset{\sim}{v}_{\hat{P}},\ {}^R\underset{\sim}{a}_{\hat{P}}$: the velocity and acceleration of $\hat{P}$ in R (recall that $\hat{P}$ is fixed on B)

$2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)$: Coriolis acceleration of P in R

Note: The ***velocity*** and ***acceleration*** of $\hat{P}$ can be determined by using the ***two-point formulae*** for points that are fixed on rigid bodies as discussed in Unit 3.

Derivation



The results shown above can be easily shown by using the "***derivative rule***" discussed in Unit 1. Consider the rigid body shown in the diagram at the right. Here,

R : is a fixed reference frame,

B : is a rigid body,

P : is a point ***moving*** on B

$\hat{P}$: is a point ***fixed*** on B that ***coincides*** with P

Q : is a point ***fixed*** on B ***not*** coincident with P

The ***velocity*** of P may be found as follows

$$\left.\begin{aligned} {}^R\underset{\sim}{v}_P &= \frac{{}^R d\,\underset{\sim}{p}}{dt} = \frac{{}^R d}{dt}\left(\underset{\sim}{q}+\underset{\sim}{r}\right) = \frac{{}^R d\,\underset{\sim}{q}}{dt} + \frac{{}^R d\,\underset{\sim}{r}}{dt} \\ &= {}^R\underset{\sim}{v}_Q + \frac{{}^B d\,\underset{\sim}{r}}{dt} + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \\ &= {}^R\underset{\sim}{v}_Q + {}^B\underset{\sim}{v}_P + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right) \end{aligned}\right\} \Rightarrow \boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P} \quad (\text{as } \underset{\sim}{r} \to \underset{\sim}{0})$$

The ***acceleration*** of P may be found as follows

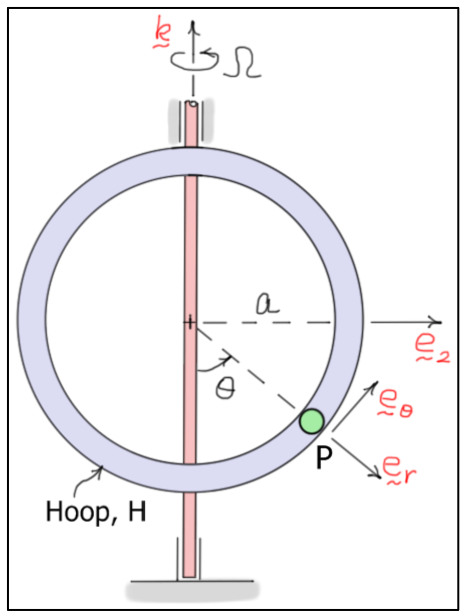
$$\begin{aligned} {}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_Q + {}^B\underset{\sim}{v}_P + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right) \\ &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_Q\right) + \left\{\frac{{}^R d}{dt}\left({}^B\underset{\sim}{v}_P\right)\right\} + \left\{\frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\} \\ &= {}^R\underset{\sim}{a}_Q + \left\{\frac{{}^B d}{dt}\left({}^B\underset{\sim}{v}_P\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left(\frac{{}^B d\,\underset{\sim}{r}}{dt} + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right)\right\} \\ &= {}^R\underset{\sim}{a}_Q + \left\{{}^B\underset{\sim}{a}_P + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\} \end{aligned}$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ gives the desired result.

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)}$$

Example 1:

The system shown consists of a ***hoop*** H which is affixed-to and rotates with the ***vertical shaft***. As the shaft rotates about its axis a ball P slides freely in the hoop. The shaft and hoop are rotating at a rate of Ω (rad/s), and the position of the ball in the hoop is given by the angle θ. The radius of the hoop is a. In the solution that follows, the following reference frames will be useful.



R: $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ (fixed frame)

H: $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k}$ (rotates with the hoop) $\left(\underset{\sim}{e}_2 \times \underset{\sim}{k} = \underset{\sim}{e}_1\right)$

P: $\underset{\sim}{e}_1, \underset{\sim}{e}_r, \underset{\sim}{e}_\theta$ (rotates with the hoop and ball) $\left(\underset{\sim}{e}_r \times \underset{\sim}{e}_\theta = \underset{\sim}{e}_1\right)$

Find:

${}^R\underset{\sim}{v}_P$ … the ***velocity*** of point P in R

${}^R\underset{\sim}{a}_P$ … the ***acceleration*** of point P in R

Solution:

As ***hoop*** *H* rotates about the ***vertical axis***, the ball *P* slides within the hoop. To find the velocity and acceleration of *P*, use the formulae for ***points moving*** on ***non-stationary bodies***.

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^H\underset{\sim}{v}_P}$$ ($\hat{P}$ is a point ***fixed on*** *H* that ***coincides*** with *P* at the instant shown)

where

$$\boxed{{}^R\underset{\sim}{v}_{\hat{P}} = -a\,S_\theta\,\Omega\,\underset{\sim}{e}_1}$$ ($\hat{P}$ has ***circular motion*** around the vertical axis; radius of the circle is "$a\,S_\theta$")

$$\boxed{{}^H\underset{\sim}{v}_P = a\,\dot{\theta}\,\underset{\sim}{e}_\theta = a\,\dot{\theta}\left(C_\theta\,\underset{\sim}{e}_2 + S_\theta\,\underset{\sim}{k}\right)}$$ (***circular motion*** of *P* on *H* in the $(\underset{\sim}{e}_2, \underset{\sim}{k})$ plane)

Adding these two results, gives the ***velocity*** of *P*.

$$\boxed{{}^R\underset{\sim}{v}_P = -a\,S_\theta\,\Omega\,\underset{\sim}{e}_1 + a\,\dot{\theta}\,C_\theta\,\underset{\sim}{e}_2 + a\,\dot{\theta}\,S_\theta\,\underset{\sim}{k}}$$

The formula for the ***acceleration*** of a point moving on a body is

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^H\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_H \times {}^H\underset{\sim}{v}_P\right)}$$

Here, $\hat{P}$ has the same meaning as before and the last term is the Coriolis acceleration.

$$\boxed{{}^R\underset{\sim}{a}_{\hat{P}} = -a\,S_\theta\,\dot{\Omega}\,\underset{\sim}{e}_1 - a\,S_\theta\,\Omega^2\,\underset{\sim}{e}_2}$$ (***circular motion*** of $\hat{P}$ in *R*)

$$\boxed{{}^H\underset{\sim}{a}_P = a\,\ddot{\theta}\,\underset{\sim}{e}_\theta - a\,\dot{\theta}^2\,\underset{\sim}{e}_r}$$ (***circular motion*** of *P* on *H*)

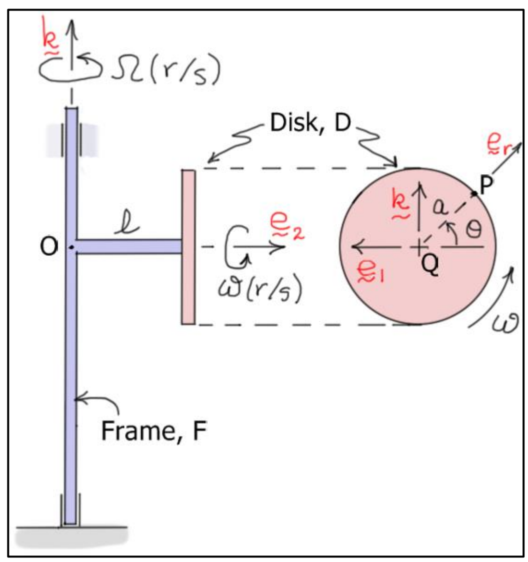
$$\boxed{2\left({}^R\underset{\sim}{\omega}_H \times {}^H\underset{\sim}{v}_P\right) = 2\Omega\,\underset{\sim}{k} \times a\,\dot{\theta}\left(C_\theta\,\underset{\sim}{e}_2 + S_\theta\,\underset{\sim}{k}\right) = -2\,a\,\dot{\theta}\,\Omega\,C_\theta\,\underset{\sim}{e}_1}$$ (***Coriolis*** acceleration)

Adding these three results, gives the acceleration of *P*.

$$\boxed{{}^R\underset{\sim}{a}_P = -\left(a\dot{\Omega}S_\theta + 2a\dot{\theta}\Omega C_\theta\right)\underset{\sim}{e}_1 + \left(a\ddot{\theta}C_\theta - a\dot{\theta}^2 S_\theta - a\Omega^2 S_\theta\right)\underset{\sim}{e}_2 + \left(a\ddot{\theta}S_\theta + a\dot{\theta}^2 C_\theta\right)\underset{\sim}{k}}$$

Example 2:

The system shown consists of two connected bodies – the frame *F* and the disk *D*. Frame *F* rotates at a rate of Ω (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk *D* is affixed to and rotates relative to *F* at a rate of ω (rad/s) about the horizontal arm of *F* (annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames:

$R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ (fixed frame)

$F:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating frame)

Find: (express the results using unit vectors fixed in F)

a) ${}^R\underset{\sim}{v}_P$ … the ***velocity*** of point P in R using the ***formula for points moving on bodies***

b) ${}^R\underset{\sim}{a}_P$ … the ***acceleration*** of point P in R using the ***formula for points moving on bodies***

Solution:

This problem has been solved in previous Units using other methods. Here the formulae for points moving on rigid bodies are used. To do this, point P is treated as a point that is moving on frame F (rather than in previous notes where P was defined to be a point fixed on the disk).

a) Using this approach, the ***velocity*** of P may be written as

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^F\underset{\sim}{v}_P}$$ (here, $\hat{P}$ is fixed on F and coincides with P at the instant shown)

The points $\hat{P}$ and O are both fixed on F, so the ***two-point formula*** for velocity can be used to find ${}^R\underset{\sim}{v}_{\hat{P}}$.

$$ {}^R\underset{\sim}{v}_{\hat{P}} = \underbrace{{}^R\underset{\sim}{v}_O}_{\text{zero}} + {}^R\underset{\sim}{v}_{\hat{P}/O} = {}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{\hat{P}/O} = \Omega\,\underset{\sim}{k} \times \left(-a C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a S_\theta\,\underset{\sim}{k}\right) \;\Rightarrow\; \boxed{{}^R\underset{\sim}{v}_{\hat{P}} = \Omega\left(-a C_\theta\,\underset{\sim}{e}_2 - \ell\,\underset{\sim}{e}_1\right)} $$

The velocity of P on F is found by holding the frame F fixed. (P has ***circular motion*** on F)

$$ {}^F\underset{\sim}{v}_P = \underbrace{{}^F\underset{\sim}{v}_Q}_{\text{zero}} + {}^F\underset{\sim}{v}_{P/Q} = {}^F\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{P/Q} = \omega\,\underset{\sim}{e}_2 \times \left(-a C_\theta\,\underset{\sim}{e}_1 + a S_\theta\,\underset{\sim}{k}\right) \;\Rightarrow\; \boxed{{}^F\underset{\sim}{v}_P = \omega\left(a C_\theta\,\underset{\sim}{k} + a S_\theta\,\underset{\sim}{e}_1\right)} $$

Adding these two results gives the velocity of P.

$$\boxed{{}^R\underset{\sim}{v}_P = \left(a\,\omega S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 + \left(-a\,\Omega C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega C_\theta\right)\underset{\sim}{k}}$$

b) Using this same approach, the ***acceleration*** of P may be written as

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^F\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_F \times {}^F\underset{\sim}{v}_P\right)}$$ (here again, $\hat{P}$ is fixed on F and coincides with P)

where, using the two-point formula for acceleration

$$ {}^R\underset{\sim}{a}_{\hat{P}} = \underbrace{{}^R\underset{\sim}{a}_O}_{\text{zero}} + {}^R\underset{\sim}{a}_{\hat{P}/O} = \left[{}^R\underset{\sim}{\alpha}_F \times \underset{\sim}{r}_{\hat{P}/O}\right] + \left[{}^R\underset{\sim}{\omega}_F \times {}^R\underset{\sim}{v}_{\hat{P}/O}\right] $$

$$ = \left[\dot{\Omega}\,\underset{\sim}{k} \times \left(-a C_\theta\,\underset{\sim}{e}_1 + \ell\,\underset{\sim}{e}_2 + a S_\theta\,\underset{\sim}{k}\right)\right] + \left[\Omega\,\underset{\sim}{k} \times \underbrace{\Omega\left(-a C_\theta\,\underset{\sim}{e}_2 - \ell\,\underset{\sim}{e}_1\right)}_{\text{from part (a)}}\right] $$

$$ = \dot{\Omega}\left(-a C_\theta\,\underset{\sim}{e}_2 - \ell\,\underset{\sim}{e}_1\right) + \Omega^2\left(a C_\theta\,\underset{\sim}{e}_1 - \ell\,\underset{\sim}{e}_2\right) $$

$$ \Rightarrow \quad \boxed{{}^R\underset{\sim}{a}_{\hat{P}} = \left(a\,\Omega^2 C_\theta - \ell\,\dot{\Omega}\right)\underset{\sim}{e}_1 - \left(\ell\,\Omega^2 + a\,\dot{\Omega} C_\theta\right)\underset{\sim}{e}_2} $$

$$
\begin{aligned}
{}^{F}\underset{\sim}{a}_{P} &= \left[{}^{F}\underset{\sim}{\alpha}_{D}\times\underset{\sim}{r}_{P/Q}\right]+\left[{}^{F}\underset{\sim}{\omega}_{D}\times{}^{F}\underset{\sim}{v}_{P/Q}\right]\\
&=\left[\dot{\omega}\,\underset{\sim}{e}_{2}\times\left(-a\,C_{\theta}\,\underset{\sim}{e}_{1}+a\,S_{\theta}\,\underset{\sim}{k}\right)\right]+\left[\omega\,\underset{\sim}{e}_{2}\times\left(\omega\left(a\,C_{\theta}\,\underset{\sim}{k}+a\,S_{\theta}\,\underset{\sim}{e}_{1}\right)\right)\right]
\end{aligned}
$$

(P has ***circular motion*** on F)

$$
=\dot{\omega}\left(a\,C_{\theta}\,\underset{\sim}{k}+a\,S_{\theta}\,\underset{\sim}{e}_{1}\right)+\omega^{2}\left(a\,C_{\theta}\,\underset{\sim}{e}_{1}-a\,S_{\theta}\,\underset{\sim}{k}\right)
$$

$$
\Rightarrow\quad \boxed{{}^{F}\underset{\sim}{a}_{P}=\left(a\,\dot{\omega}S_{\theta}+a\,\omega^{2}C_{\theta}\right)\underset{\sim}{e}_{1}+\left(a\,\dot{\omega}C_{\theta}-a\,\omega^{2}\,S_{\theta}\right)\underset{\sim}{k}}
$$

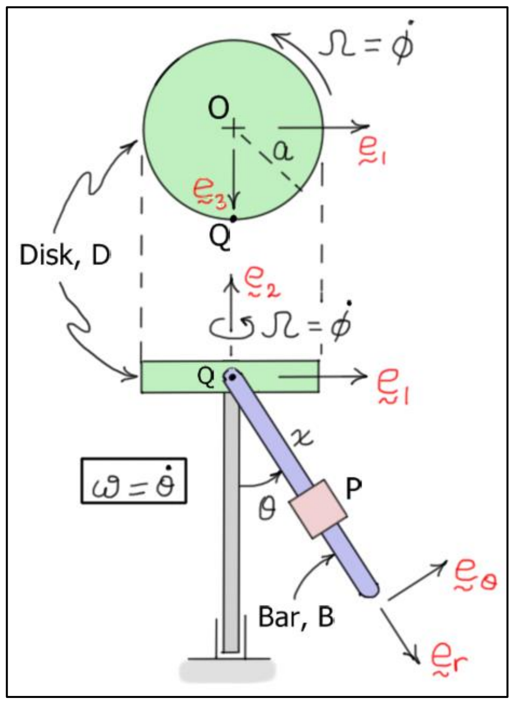
$$
\boxed{2\left({}^{R}\underset{\sim}{\omega}_{F}\times{}^{F}\underset{\sim}{v}_{P}\right)=2\Omega\,\underset{\sim}{k}\times\omega\left(a\,C_{\theta}\,\underset{\sim}{k}+a\,S_{\theta}\,\underset{\sim}{e}_{1}\right)=2\,a\,\omega\Omega S_{\theta}\,\underset{\sim}{e}_{2}}
$$

(***Coriolis*** acceleration)

Adding these three results gives the final result.

$$
\boxed{\begin{aligned}
{}^{R}\underset{\sim}{a}_{P} &=\left(a\,\dot{\omega}\,S_{\theta}-\ell\,\dot{\Omega}+a\,C_{\theta}\left(\omega^{2}+\Omega^{2}\right)\right)\underset{\sim}{e}_{1}+\left(2\,a\,\omega\Omega\,S_{\theta}-a\,\dot{\Omega}\,C_{\theta}-\ell\,\Omega^{2}\right)\underset{\sim}{e}_{2}\\
&\quad+\left(a\,\dot{\omega}\,C_{\theta}-a\,\omega^{2}S_{\theta}\right)\underset{\sim}{k}
\end{aligned}}
$$

Example 3: Slider on a Rotating Bar



The figure shows the ***top*** and ***side*** views of a system which consists of a disk D, bar B, and a slider P. The disk is affixed-to and rotates with a shaft about a vertical axis (direction annotated by unit vector $\underset{\sim}{e}_{2}$). The angle of rotation is given by the variable ϕ. The bar is pinned to the ***front edge*** of the disk and swings freely about the axis of the pin (along the $\underset{\sim}{e}_{3}$ direction). The ***angle*** of the bar relative to the ***vertical*** is θ. The slider slides along the bar. It's position relative to the pivot point Q is given by the variable x. As shown, P could slide off the end of the bar, but it could easily be held on the bar using a spring.

Reference frames:

$D:(\underset{\sim}{e}_{1},\underset{\sim}{e}_{2},\underset{\sim}{e}_{3})$

$B:(\underset{\sim}{e}_{r},\underset{\sim}{e}_{\theta},\underset{\sim}{e}_{3})$

Find:

a) ${}^{R}\underset{\sim}{v}_{P}$ the ***velocity*** of P in a fixed frame R

b) ${}^{R}\underset{\sim}{a}_{P}$ the ***acceleration*** of P in R

Solution:

a) To find the ***velocity*** of P as it moves on the bar, the ***angular velocity*** of the bar is needed. Using the ***summation rule*** for angular velocities, ${}^{R}\underset{\sim}{\omega}_{B}$ the angular velocity of B relative to the ground frame may be written as

$$
\boxed{{}^{R}\underset{\sim}{\omega}_{B}={}^{R}\underset{\sim}{\omega}_{D}+{}^{D}\underset{\sim}{\omega}_{B}=\Omega\,\underset{\sim}{e}_{2}+\omega\,\underset{\sim}{e}_{3}}
$$

Using the formula for points moving on bodies, the velocity of P may now be written as

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P} \qquad (P \text{ moves on } B;\ \hat{P} \text{ is fixed on } B \text{ and coincides with } P)$$

where, using the two-point formula for velocity

$$\begin{aligned} {}^R\underset{\sim}{v}_{\hat{P}} &= {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{v}_{\hat{P}/Q} \\ &= a\Omega\underset{\sim}{e}_1 + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{\hat{P}/Q}\right) \\ &= a\Omega\underset{\sim}{e}_1 + \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ xS_\theta & -xC_\theta & 0 \end{vmatrix} \end{aligned} \quad \Rightarrow \quad \boxed{{}^R\underset{\sim}{v}_{\hat{P}} = (a\Omega + x\omega C_\theta)\,\underset{\sim}{e}_1 + (x\omega S_\theta)\,\underset{\sim}{e}_2 - (x\Omega S_\theta)\,\underset{\sim}{e}_3}$$

$$\boxed{{}^B\underset{\sim}{v}_P = \dot{x}\,\underset{\sim}{e}_r = \dot{x}(S_\theta\underset{\sim}{e}_1 - C_\theta\underset{\sim}{e}_2)} \quad (P \text{ has } \textbf{\textit{rectilinear motion}} \text{ on } B)$$

Adding the two previous equations gives the final result.

$$\boxed{{}^R\underset{\sim}{v}_P = (a\Omega + x\omega C_\theta + \dot{x}S_\theta)\,\underset{\sim}{e}_1 + (x\omega S_\theta - \dot{x}C_\theta)\,\underset{\sim}{e}_2 - (x\Omega S_\theta)\,\underset{\sim}{e}_3}$$

b) To find the ***acceleration*** of P as it moves on the bar, the ***angular acceleration*** of the bar is also needed. Recall that the angular acceleration is found by ***differentiating*** the angular velocity vector.

$${}^R\underset{\sim}{\alpha}_B = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \dot{\Omega}\underset{\sim}{e}_2 + \Omega\underbrace{\left(\frac{{}^Rd\underset{\sim}{e}_2}{dt}\right)}_{\text{zero}} + \dot{\omega}\underset{\sim}{e}_3 + \omega\left(\frac{{}^Rd\underset{\sim}{e}_3}{dt}\right) = \dot{\Omega}\underset{\sim}{e}_2 + \dot{\omega}\underset{\sim}{e}_3 + \omega\left(\Omega\underset{\sim}{e}_2 \times \underset{\sim}{e}_3\right)$$

$$\Rightarrow \quad \boxed{{}^R\underset{\sim}{\alpha}_B = \omega\,\Omega\,\underset{\sim}{e}_1 + \dot{\Omega}\,\underset{\sim}{e}_2 + \dot{\omega}\,\underset{\sim}{e}_3}$$

Using the formulae for points moving on bodies, the acceleration of P may now be written as

$$\boxed{\begin{aligned} {}^R\underset{\sim}{a}_P &= {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) \\ &= {}^R\underset{\sim}{a}_Q + {}^R\underset{\sim}{a}_{\hat{P}/Q} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) \end{aligned}}$$

where

$$\boxed{{}^R\underset{\sim}{a}_Q = a\dot{\Omega}\underset{\sim}{e}_1 - a\Omega^2\underset{\sim}{e}_3} \quad (Q \text{ has } \textbf{\textit{circular motion}} \text{ about } O \text{ in } R)$$

$$\begin{aligned} {}^R\underset{\sim}{a}_{\hat{P}/Q} &= ({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{\hat{P}/Q}) + ({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{\hat{P}/Q}) \\ &= \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \omega\Omega & \dot{\Omega} & \dot{\omega} \\ xS_\theta & -xC_\theta & 0 \end{vmatrix} + \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ x\omega C_\theta & x\omega S_\theta & -x\Omega S_\theta \end{vmatrix} \end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned} {}^R\underset{\sim}{a}_{\hat{P}/Q} = &\left(x\dot{\omega}C_\theta - x\Omega^2 S_\theta - x\omega^2 S_\theta\right)\underset{\sim}{e}_1 + \left(x\dot{\omega}S_\theta + x\omega^2 C_\theta\right)\underset{\sim}{e}_2 - \\ &\left(x\omega\Omega C_\theta + x\dot{\Omega}S_\theta + x\omega\Omega C_\theta\right)\underset{\sim}{e}_3 \end{aligned}} \quad (\hat{P} \text{ and } Q \text{ are both } \textbf{\textit{fixed}} \text{ on } B)$$

$$\boxed{{}^{B}\underset{\sim}{a}_P = \ddot{x}\,\underset{\sim}{e}_r = \ddot{x}\left(S_\theta\,\underset{\sim}{e}_1 - C_\theta\,\underset{\sim}{e}_2\right)}$$ (P has ***rectilinear motion*** on B)

$$\boxed{2\,{}^{R}\underset{\sim}{\omega}_B \times {}^{B}\underset{\sim}{v}_P = 2\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ \dot{x}S_\theta & -\dot{x}C_\theta & 0 \end{vmatrix} = 2\left[\left(\dot{x}\,\omega C_\theta\right)\underset{\sim}{e}_1 + \left(\dot{x}\,\omega S_\theta\right)\underset{\sim}{e}_2 - \left(\dot{x}\,\Omega S_\theta\right)\underset{\sim}{e}_3\right]}$$ (***Coriolis*** acceleration)

Adding the four terms gives the final result.

$$\boxed{\begin{aligned}{}^{R}\underset{\sim}{a}_P = &\left(a\dot{\Omega} + x\dot{\omega}C_\theta - xS_\theta(\omega^2+\Omega^2) + \ddot{x}S_\theta + 2\dot{x}\,\omega C_\theta\right)\underset{\sim}{e}_1 + \left(x\dot{\omega}S_\theta + x\omega^2 C_\theta - \ddot{x}C_\theta + 2\dot{x}\,\omega S_\theta\right)\underset{\sim}{e}_2 - \\ &\left(a\Omega^2 + x\dot{\Omega}S_\theta + 2x\,\omega\Omega C_\theta + 2\dot{x}\,\Omega S_\theta\right)\underset{\sim}{e}_3\end{aligned}}$$

Note that in the solution presented for this example, the ***velocity*** and ***acceleration*** of point $\hat{P}$ were calculated using the ***two-point formulae*** for points fixed on bodies. As shown in Example 2, the ***formulae for points moving on bodies*** could also be used. See Example 4 for a solution that ***extends*** the application of this latter approach.

Example 4: Slider on a Rotating Bar Revisited

In this example the velocity and acceleration of the slider in Example 3 are calculated by ***recursively*** applying the ***formulae for points moving on bodies***.

Solution:

a) Using the formula for points moving on bodies, the ***velocity*** of P may now be written as

$$\boxed{\begin{aligned}{}^{R}\underset{\sim}{v}_P &= {}^{R}\underset{\sim}{v}_{\hat{P}} + {}^{B}\underset{\sim}{v}_P \\ &= {}^{R}\underset{\sim}{v}_{\hat{\hat{P}}} + {}^{D}\underset{\sim}{v}_{\hat{P}} + {}^{B}\underset{\sim}{v}_P\end{aligned}}$$

(P moves on B; $\hat{P}$ is fixed on B and moves on D; $\hat{\hat{P}}$ is fixed on D)

(points P, $\hat{P}$, and $\hat{\hat{P}}$ all coincide)

where

$$\boxed{{}^{R}\underset{\sim}{v}_{\hat{\hat{P}}} = \underbrace{{}^{R}\underset{\sim}{v}_O}_{\text{zero}} + {}^{R}\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{\hat{\hat{P}}/O} = \Omega\,\underset{\sim}{e}_2 \times \left(xS_\theta\,\underset{\sim}{e}_1 - xC_\theta\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_3\right) = \left(a\,\Omega\right)\underset{\sim}{e}_1 - \left(x\,\Omega S_\theta\right)\underset{\sim}{e}_3}$$ (***circular motion***)

$$\boxed{{}^{D}\underset{\sim}{v}_{\hat{P}} = x\,\omega\,\underset{\sim}{e}_\theta = x\,\omega\left(C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{e}_2\right)}$$ (***circular motion*** of $\hat{P}$ around Q)

$$\boxed{{}^{B}\underset{\sim}{v}_P = \dot{x}\,\underset{\sim}{e}_r = \dot{x}\left(S_\theta\,\underset{\sim}{e}_1 - C_\theta\,\underset{\sim}{e}_2\right)}$$ (***rectilinear motion*** of P on B)

Adding the three previous results gives the final result

$$\boxed{{}^{R}\underset{\sim}{v}_P = (a\,\Omega + x\,\omega C_\theta + \dot{x}S_\theta)\,\underset{\sim}{e}_1 + (x\,\omega S_\theta - \dot{x}C_\theta)\,\underset{\sim}{e}_2 - (x\,\Omega S_\theta)\,\underset{\sim}{e}_3}$$

b) Using the formula for points moving on bodies, the ***acceleration*** of P may now be written as

$$\begin{aligned} {}^R\underset{\sim}{a}_P &= {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) \\ &= {}^R\underset{\sim}{a}_{\hat{\hat{P}}} + {}^D\underset{\sim}{a}_{\hat{P}} + 2\left({}^R\underset{\sim}{\omega}_D \times {}^D\underset{\sim}{v}_{\hat{P}}\right) + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) \end{aligned}$$

(points P, $\hat{P}$, and $\hat{\hat{P}}$ as previously defined)

where

$${}^R\underset{\sim}{a}_{\hat{\hat{P}}} = \underbrace{{}^R\underset{\sim}{a}_O}_{\text{zero}} + \left({}^R\underset{\sim}{\alpha}_D \times \underset{\sim}{r}_{\hat{\hat{P}}/O}\right) + \left({}^R\underset{\sim}{\omega}_D \times {}^R\underset{\sim}{v}_{\hat{\hat{P}}}\right)$$

$$= \left[\dot{\Omega}\,\underset{\sim}{e}_2 \times \left(x S_\theta\,\underset{\sim}{e}_1 - x C_\theta\,\underset{\sim}{e}_2 + a\,\underset{\sim}{e}_3\right)\right] + \left[\Omega\,\underset{\sim}{e}_2 \times \left(a\,\Omega\,\underset{\sim}{e}_1 - x\,\Omega\,S_\theta\,\underset{\sim}{e}_3\right)\right]$$

$\left\{\hat{\hat{P}}\right.$ has ***circular motion*** in R

$$\Rightarrow \boxed{{}^R\underset{\sim}{a}_{\hat{\hat{P}}} = \left(a\dot{\Omega} - x\Omega^2 S_\theta\right)\underset{\sim}{e}_1 - \left(x\dot{\Omega}S_\theta + a\Omega^2\right)\underset{\sim}{e}_3}$$

$${}^D\underset{\sim}{a}_{\hat{P}} = \left(x\dot{\omega}\right)\underset{\sim}{e}_\theta - \left(x\omega^2\right)\underset{\sim}{e}_r = \left(x\dot{\omega}\right)\left(C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{e}_2\right) - \left(x\omega^2\right)\left(S_\theta\,\underset{\sim}{e}_1 - C_\theta\,\underset{\sim}{e}_2\right)$$

$$\Rightarrow \boxed{{}^D\underset{\sim}{a}_{\hat{P}} = \left(x\dot{\omega}C_\theta - x\omega^2 S_\theta\right)\underset{\sim}{e}_1 + \left(x\dot{\omega}S_\theta + x\omega^2 C_\theta\right)\underset{\sim}{e}_2}$$

$\left\{\hat{P}\right.$ has ***circular motion*** on D

$$2\left({}^R\underset{\sim}{\omega}_D \times {}^D\underset{\sim}{v}_{\hat{P}}\right) = 2\left(\Omega\,\underset{\sim}{e}_2\right) \times x\,\omega\left(C_\theta\,\underset{\sim}{e}_1 + S_\theta\,\underset{\sim}{e}_2\right) = -2\,x\,\omega\,\Omega\,\underset{\sim}{e}_3$$

(***Coriolis*** acceleration)

$${}^B\underset{\sim}{a}_P = \ddot{x}\,\underset{\sim}{e}_r = \ddot{x}\left(S_\theta\,\underset{\sim}{e}_1 - C_\theta\,\underset{\sim}{e}_2\right)$$

(P has ***rectilinear motion*** on B)

$$2\,{}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P = 2\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ \dot{x}S_\theta & -\dot{x}C_\theta & 0 \end{vmatrix} = 2\left[\left(\dot{x}\,\omega\,C_\theta\right)\underset{\sim}{e}_1 + \left(\dot{x}\,\omega\,S_\theta\right)\underset{\sim}{e}_2 - \left(\dot{x}\,\Omega\,S_\theta\right)\underset{\sim}{e}_3\right]$$

(***Coriolis*** acceleration)

Adding the previous five terms gives the final result.

$$\begin{aligned} {}^R\underset{\sim}{a}_P = {} & \left(a\dot{\Omega} + x\dot{\omega}C_\theta - xS_\theta(\omega^2 + \Omega^2) + \ddot{x}S_\theta + 2\dot{x}\,\omega\,C_\theta\right)\underset{\sim}{e}_1 + \left(x\dot{\omega}S_\theta + x\omega^2 C_\theta - \ddot{x}C_\theta + 2\dot{x}\,\omega\,S_\theta\right)\underset{\sim}{e}_2 - \\ & \left(a\Omega^2 + x\dot{\Omega}S_\theta + 2\,x\,\omega\,\Omega C_\theta + 2\dot{x}\,\Omega\,S_\theta\right)\underset{\sim}{e}_3 \end{aligned}$$

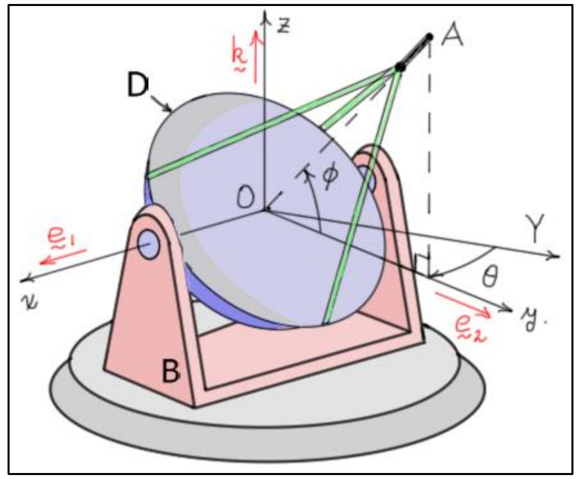
Compare this approach to that taken in Example 3.

Notes

1. The ***formulae for points fixed on bodies*** together with the ***formulae for points moving on bodies*** may be used ***recursively*** to compute the velocities and accelerations of ***remote*** points in a multibody system.
2. Using this approach, calculations of velocities and accelerations ***do not require differentiation***. It requires only ***vector multiplication*** and ***addition***. Results may be calculated that are valid for ***all time*** or only a ***specific instant*** of time. Recall that when using ***direct differentiation***, the method requires the results be valid for all time. As always, differentiation is still required to compute angular acceleration vectors.

Exercises:

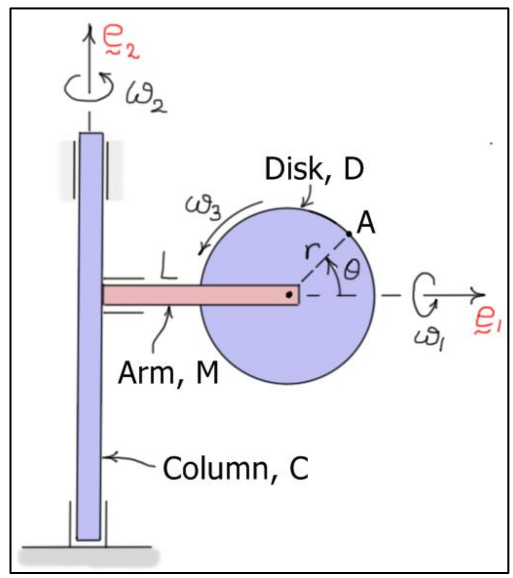
4.1 The antenna system shown has two components, the base *B* and the antenna dish *D*. Base *B* rotates relative to the ground about the fixed *z*-axis, and dish *D* rotates relative to *B* about the rotating *x*-axis. At any instant, the angle between the *y*-axis ($\underset{\sim}{e}_2$) and the fixed *Y*-axis is θ, and the angle between line *OA* and the *y*-axis is ϕ. Given values for θ, ϕ, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point *A* in a fixed frame *R* using the ***formulae for points moving on bodies***.



Answers: $\underset{\sim}{v}_A = L\left(\dot{\theta}C_\phi\,\underset{\sim}{e}_1 - \dot{\phi}S_\phi\,\underset{\sim}{e}_2 + \dot{\phi}C_\phi\,\underset{\sim}{k}\right)$ (results expressed in $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$)

$$\underset{\sim}{a}_A = L\left(\left(\ddot{\theta}C_\phi - 2\dot{\theta}\dot{\phi}S_\phi\right)\underset{\sim}{e}_1 - \left(\ddot{\phi}S_\phi + \dot{\phi}^2C_\phi + \dot{\theta}^2C_\phi\right)\underset{\sim}{e}_2 + \left(\ddot{\phi}C_\phi - \dot{\phi}^2S_\phi\right)\underset{\sim}{k}\right)$$

4.2 The system shown has three components, a vertical column *C*, a horizontal arm *M*, and a disk *D*. The disk rotates relative to the arm at a rate ω_3 (rad/sec) about the $\underset{\sim}{n}_3$ direction (normal to *D*), the arm rotates relative to the column at a rate of ω_1 (rad/sec) about the $\underset{\sim}{e}_1$ direction, and the column rotates relative to the ground at a rate of ω_2 (rad/sec) about the fixed $\underset{\sim}{e}_2$ direction. The unit vector set $C:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is fixed in the column, and the unit vector set $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ is fixed in arm M. Given values for ω_1, ω_2, ω_3, and their time derivatives, find ${}^R\underset{\sim}{v}_A$ and ${}^R\underset{\sim}{a}_A$ the velocity and acceleration of point *A* in a fixed frame *R* using the ***formulae for points moving on bodies***.



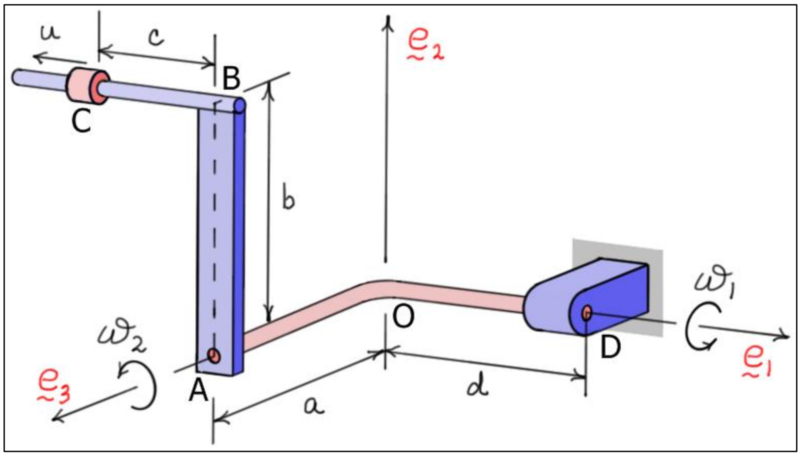
Answers: (expressed in $M:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$)

$${}^R\underset{\sim}{v}_A = r(\omega_2S_\theta S_\phi - \omega_3S_\theta)\underset{\sim}{e}_1 + (r\omega_3C_\theta - (L+rC_\theta)\omega_2S_\phi)\underset{\sim}{n}_2 + (r\omega_1S_\theta - (L+rC_\theta)\omega_2C_\phi)\underset{\sim}{n}_3$$

$$\begin{aligned}{}^R\underset{\sim}{a}_A = {} & (r\dot{\omega}_2S_\theta S_\phi - r\dot{\omega}_3S_\theta - r\omega_3^2C_\theta - (L+rC_\theta)\omega_2^2 + 2r\omega_1\omega_2S_\theta C_\phi + 2r\omega_2\omega_3S_\phi C_\theta)\underset{\sim}{e}_1 + \\ & (r\dot{\omega}_3C_\theta - (L+rC_\theta)\dot{\omega}_2S_\phi - r\omega_1^2S_\theta - r\omega_2^2S_\theta S_\phi^2 - r\omega_3^2S_\theta + 2r\omega_2\omega_3S_\theta S_\phi)\underset{\sim}{n}_2 + \\ & (r\dot{\omega}_1S_\theta - (L+rC_\theta)\dot{\omega}_2C_\phi - r\omega_2^2S_\theta S_\phi C_\phi + 2r\omega_2\omega_3S_\theta C_\phi + 2r\omega_1\omega_3C_\theta)\underset{\sim}{n}_3\end{aligned}$$

<u>Hint</u>: Here ϕ is the angle between the plane of the disk and the $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ plane $\left(\dot{\phi} = \omega_1\right)$. The diagram shows the position where $\phi = 0$.

4.3 The system shown consists of two bodies and a collar. The bent bar *DOA* is connected to the ground by a simple revolute joint at *D* which allows rotation about the fixed $\underset{\sim}{e}_1$ direction only. Body *ABC* is connected to *DOA* with a simple revolute joint which allows rotation of *ABC* relative to *DOA* only about the $\underset{\sim}{e}_3$ direction. The collar at *C* is traveling along body *ABC* at a speed of $\dot{c} = u$.



Calculate ${}^R\underset{\sim}{v}_C$ and ${}^R\underset{\sim}{a}_C$ the velocity and acceleration of *C* relative to the ground ***at the instant shown***.

Answers: $${}^R\underset{\sim}{v}_C = -\left(u + b\,\omega_2\right)\underset{\sim}{e}_1 - \left(a\,\omega_1 + c\,\omega_2\right)\underset{\sim}{e}_2 + \left(b\,\omega_1\right)\underset{\sim}{e}_3$$ (at the instant shown)

$${}^R\underset{\sim}{a}_C = \left(c\,\omega_2^2 - b\,\dot{\omega}_2 - \dot{u}\right)\underset{\sim}{e}_1 - \left(a\,\dot{\omega}_1 + c\,\dot{\omega}_2 + b\,\omega_1^2 + b\,\omega_2^2 + 2u\,\omega_2\right)\underset{\sim}{e}_2 + \left(b\,\dot{\omega}_1 - a\,\omega_1^2 - 2c\,\omega_1\omega_2\right)\underset{\sim}{e}_3$$ (at the instant shown)

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