An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 7

Application of Concepts to Systems with Complex Interconnecting Joints

Summary

The ***basic kinematical concepts*** for mechanical systems were presented in Units 1 through 4. The concepts were applied to systems in which the ***relative orientation changes*** between adjoining bodies were described by a ***single rotation*** - simple revolute joints. In Units 5 and 6 concepts were presented for describing the relative orientation changes between ***any two adjoining bodies*** by using a set of up to three ***orientation angles*** or a set of four ***Euler parameters***. This unit ***applies*** the concepts from Units 1 through 6 to analyze the kinematics of systems with ***more complex connecting joints***. The major differences in applying the kinematical concepts of Units 1 through 4 are in the calculations of orientation, angular velocity and angular acceleration.

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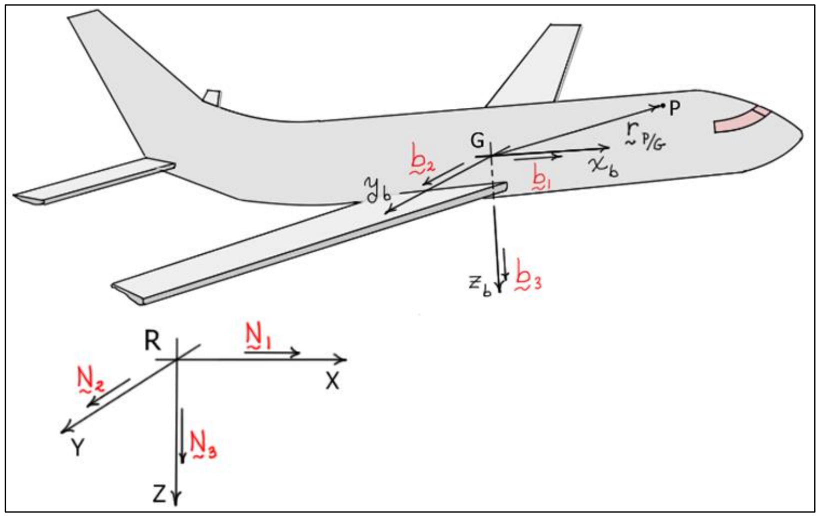
Example 1:

The orientation of an aircraft relative to reference frame R is defined by a 3-2-1 ***body-fixed*** rotation sequence (ψ,θ,ϕ). The ***velocity*** of the aircraft is given in body-fixed directions as ${}^R\underset{\sim}{v}_G = u\,\underset{\sim}{b}_1 + v\,\underset{\sim}{b}_2 + w\,\underset{\sim}{b}_3$. The ***position*** of a point P on the aircraft relative to the mass center G is given as $\underset{\sim}{r}_{P/G} = p_1\,\underset{\sim}{b}_1 + p_2\,\underset{\sim}{b}_2 + p_3\,\underset{\sim}{b}_3$. Point P is assumed to be ***fixed*** on the aircraft.



Find:

a) the components of ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P in R in both the ***body-fixed*** and ***base*** frames

b) the components of ${}^R\underset{\sim}{a}_P$ the ***acceleration*** of P in R in both the ***body-fixed*** and ***base*** frames

Solution:

a) Using the method of ***direct differentiation*** as presented in Unit 2, ${}^R\underset{\sim}{v}_P$ can be found as follows

$$ {}^R\underset{\sim}{v}_P = \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_P\right) = \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_G + \underset{\sim}{r}_{P/G}\right) = \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_G\right) + \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_{P/G}\right) = {}^R\underset{\sim}{v}_G + \underbrace{\frac{{}^Bd}{dt}\left(\underset{\sim}{r}_{P/G}\right)}_{\text{zero}} + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right) $$

$$ = \left(u\,\underset{\sim}{b}_1 + v\,\underset{\sim}{b}_2 + w\,\underset{\sim}{b}_3\right) + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ p_1 & p_2 & p_3 \end{vmatrix} $$

$$ \Rightarrow \boxed{ {}^R\underset{\sim}{v}_P = \underbrace{\left(u + \omega_2 p_3 - \omega_3 p_2\right)}_{v_1}\underset{\sim}{b}_1 + \underbrace{\left(v + \omega_3 p_1 - \omega_1 p_3\right)}_{v_2}\underset{\sim}{b}_2 + \underbrace{\left(w + \omega_1 p_2 - \omega_2 p_1\right)}_{v_3}\underset{\sim}{b}_3 } $$

Here (as presented in Unit 5): $\boxed{\omega_1 = \dot{\phi} - \dot{\psi} S_\theta}$, $\boxed{\omega_2 = \dot{\theta} C_\phi + \dot{\psi} C_\theta S_\phi}$, $\boxed{\omega_3 = -\dot{\theta} S_\phi + \dot{\psi} C_\theta C_\phi}$.

The same results can also be found by using the concepts for points ***fixed on bodies*** as presented in Unit 3. Using the ***body-fixed*** components of the ***angular velocity*** of the aircraft developed in Unit 5, the velocity of P may be written as

$$ {}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{P/G} = {}^R\underset{\sim}{v}_G + \underbrace{\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)}_{{}^R\underset{\sim}{v}_{P/G}} = \left(u\,\underset{\sim}{b}_1 + v\,\underset{\sim}{b}_2 + w\,\underset{\sim}{b}_3\right) + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ p_1 & p_2 & p_3 \end{vmatrix} $$

This clearly produces the ***same result*** as above.

The components (V_i) of ${}^R\underset{\sim}{v}_P$ in the ***base frame*** R may be found from the ***body-fixed*** components $v_i\ (i=1,2,3)$ (as defined above) using the coordinate transformation matrix for a 3-2-1 orientation angle sequence derived in Unit 5.

$$\boxed{\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = [R]^T \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\ C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\theta S_\phi \\ C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\theta C_\phi \end{bmatrix}^T \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}}$$

b) Using the method of ***direct differentiation*** (along with the derivative rule) as presented in Unit 2, ${}^R\underset{\sim}{a}_P$ can be found as follows

$$\begin{aligned}
{}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_P\right) = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_P\right) = \frac{{}^B d}{dt}\left({}^R\underset{\sim}{v}_P\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_P\right) \\
&= \left(\dot{v}_1 \underset{\sim}{b}_1 + \dot{v}_2 \underset{\sim}{b}_2 + \dot{v}_3 \underset{\sim}{b}_3\right) + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \left(\dot{u} + \dot{\omega}_2 p_3 - \dot{\omega}_3 p_2 + \omega_2 v_3 - \omega_3 v_2\right)\underset{\sim}{b}_1 + \left(\dot{v} + \dot{\omega}_3 p_1 - \dot{\omega}_1 p_3 + \omega_3 v_1 - \omega_1 v_3\right)\underset{\sim}{b}_2 \\
&\quad + \left(\dot{w} + \dot{\omega}_1 p_2 - \dot{\omega}_2 p_1 + \omega_1 v_2 - \omega_2 v_1\right)\underset{\sim}{b}_3 \\
&= \Big(\dot{u} + \alpha_2 p_3 - \alpha_3 p_2 + \omega_2\left(w + \omega_1 p_2 - \omega_2 p_1\right) - \omega_3\left(v + \omega_3 p_1 - \omega_1 p_3\right)\Big)\underset{\sim}{b}_1 \\
&\quad + \Big(\dot{v} + \alpha_3 p_1 - \alpha_1 p_3 + \omega_3\left(u + \omega_2 p_3 - \omega_3 p_2\right) - \omega_1\left(w + \omega_1 p_2 - \omega_2 p_1\right)\Big)\underset{\sim}{b}_2 \\
&\quad + \Big(\dot{w} + \alpha_1 p_2 - \alpha_2 p_1 + \omega_1\left(v + \omega_3 p_1 - \omega_1 p_3\right) - \omega_2\left(u + \omega_2 p_3 - \omega_3 p_2\right)\Big)\underset{\sim}{b}_3
\end{aligned}$$

Or

$$\boxed{{}^R\underset{\sim}{a}_P = a_1 \underset{\sim}{b}_1 + a_2 \underset{\sim}{b}_2 + a_3 \underset{\sim}{b}_3}$$

where

$$\boxed{\begin{aligned}
a_1 &= \dot{u} + \alpha_2 p_3 - \alpha_3 p_2 + \omega_2\left(w + \omega_1 p_2 - \omega_2 p_1\right) - \omega_3\left(v + \omega_3 p_1 - \omega_1 p_3\right) \\
a_2 &= \dot{v} + \alpha_3 p_1 - \alpha_1 p_3 + \omega_3\left(u + \omega_2 p_3 - \omega_3 p_2\right) - \omega_1\left(w + \omega_1 p_2 - \omega_2 p_1\right) \\
a_3 &= \dot{w} + \alpha_1 p_2 - \alpha_2 p_1 + \omega_1\left(v + \omega_3 p_1 - \omega_1 p_3\right) - \omega_2\left(u + \omega_2 p_3 - \omega_3 p_2\right)
\end{aligned}}$$

This result can also be found using the concepts for ***points fixed*** on bodies as presented in Unit 3. Using the ***body-fixed*** components of the ***angular velocity*** and ***angular acceleration*** vectors of the aircraft developed in Unit 5, the ***acceleration*** of P may be written as

$${}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_G + {}^R\underset{\sim}{a}_{P/G} = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_G\right) + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/G}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{P/G}\right)$$

Using the ***derivative rule*** for the first term gives

$$
{}^R\underset{\sim}{a}_P = \frac{{}^B d}{dt}\left({}^R\underset{\sim}{v}_G\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_G\right) + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_{P/G}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_{P/G}\right)
$$

$$
= \left(\dot{u}\,\underset{\sim}{b}_1 + \dot{v}\,\underset{\sim}{b}_2 + \dot{w}\,\underset{\sim}{b}_3\right) + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ u & v & w \end{vmatrix} + \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ p_1 & p_2 & p_3 \end{vmatrix}
$$

$$
+ \begin{vmatrix} \underset{\sim}{b}_1 & \underset{\sim}{b}_2 & \underset{\sim}{b}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ (\omega_2 p_3 - \omega_3 p_2) & (\omega_3 p_1 - \omega_1 p_3) & (\omega_1 p_2 - \omega_2 p_1) \end{vmatrix}
$$

$$
\Rightarrow \boxed{{}^R\underset{\sim}{a}_P = a_1\,\underset{\sim}{b}_1 + a_2\,\underset{\sim}{b}_2 + a_3\,\underset{\sim}{b}_3}
$$

Expanding the determinants and combining terms gives the same components $a_i\ (i=1,2,3)$ found above using direct differentiation. The angular velocity components $\omega_i\ (i=1,2,3)$ are as defined in part (a), and the angular acceleration components $\alpha_i\ (i=1,2,3)$ are as computed in Unit 5.

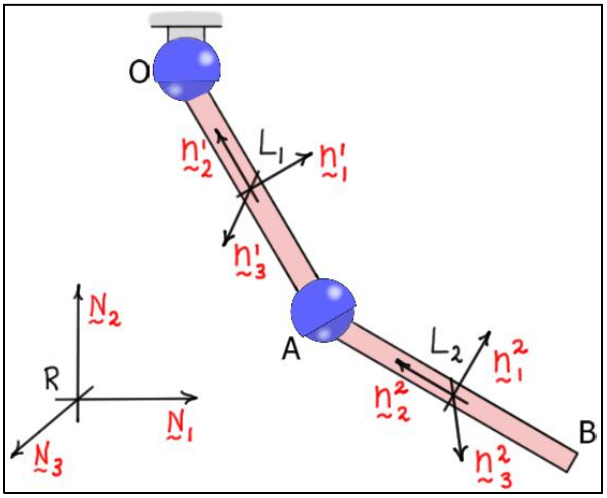
$$
\boxed{\begin{aligned}
\alpha_1 &= \dot{\omega}_1 = \ddot{\phi} - \ddot{\psi}\,S_\theta - \dot{\psi}\,\dot{\theta}\,C_\theta \\
\alpha_2 &= \dot{\omega}_2 = \ddot{\theta}\,C_\phi - \dot{\theta}\,\dot{\phi}\,S_\phi + \ddot{\psi}\,C_\theta\,S_\phi - \dot{\psi}\,\dot{\theta}\,S_\theta\,S_\phi + \dot{\psi}\,\dot{\phi}\,C_\theta\,C_\phi \\
\alpha_3 &= \dot{\omega}_3 = -\ddot{\theta}\,S_\phi - \dot{\theta}\,\dot{\phi}\,C_\phi + \ddot{\psi}\,C_\theta\,C_\phi - \dot{\psi}\,\dot{\theta}\,S_\theta\,C_\phi - \dot{\psi}\,\dot{\phi}\,C_\theta\,S_\phi
\end{aligned}}
$$

The components (A_i) of ${}^R\underset{\sim}{a}_P$ in the ***base frame*** *R* may be found from the ***body-fixed*** components (a_i as defined above) using the ***coordinate transformation matrix***.

$$
\boxed{\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = [R]^T \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\ C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\theta S_\phi \\ C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\theta C_\phi \end{bmatrix}^T \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}}
$$

Example 2:

The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ball and socket joints at O and A. The ground frame is $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ and the link frames are $L_i:\left(\underset{\sim}{n}_1^i,\underset{\sim}{n}_2^i,\underset{\sim}{n}_3^i\right)$ $(i=1,2)$. The ***orientation*** of each link is defined relative to R using a 3-1-3 ***body-fixed*** rotation sequence. The ***lengths*** of the links are ℓ_1 and ℓ_2.



Find:

a) the components of ${}^R\underset{\sim}{v}_B$ the ***velocity*** of B in R in the ***base*** frame

b) the components of ${}^R\underset{\sim}{a}_B$ the ***acceleration*** of B in R in the ***base*** frame

Solution:

a) Using the method of ***direct differentiation*** (along with the derivative rule) as presented in Unit 2, the velocity ${}^R\underset{\sim}{v}_P$ can be found as follows

$$
\begin{aligned}
{}^R\underset{\sim}{v}_B &= \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_B\right)=\frac{{}^Rd}{dt}\left(\underset{\sim}{r}_{A/O}+\underset{\sim}{r}_{B/A}\right)=\frac{{}^Rd}{dt}\left(-\ell_1\underset{\sim}{n}_2^1\right)+\frac{{}^Rd}{dt}\left(-\ell_2\underset{\sim}{n}_2^2\right)\\
&=\underbrace{\frac{{}^{L_1}d}{dt}\left(-\ell_1\underset{\sim}{n}_2^1\right)}_{\text{zero}}+\left({}^R\underset{\sim}{\omega}_{L_1}\times-\ell_1\underset{\sim}{n}_2^1\right)+\underbrace{\frac{{}^{L_2}d}{dt}\left(-\ell_2\underset{\sim}{n}_2^2\right)}_{\text{zero}}+\left({}^R\underset{\sim}{\omega}_{L_2}\times-\ell_2\underset{\sim}{n}_2^2\right)\\
&=\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ 0 & -\ell_1 & 0\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2\\ \omega_{21} & \omega_{22} & \omega_{23}\\ 0 & -\ell_2 & 0\end{vmatrix}\\
&=\left(\left(\ell_1\omega_{13}\right)\underset{\sim}{n}_1^1+(0)\underset{\sim}{n}_2^1+\left(-\ell_1\omega_{11}\right)\underset{\sim}{n}_3^1\right)+\left(\left(\ell_2\omega_{23}\right)\underset{\sim}{n}_1^2+(0)\underset{\sim}{n}_2^2+\left(-\ell_2\omega_{21}\right)\underset{\sim}{n}_3^2\right)\\
&=\begin{bmatrix}\ell_1\omega_{13} & 0 & -\ell_1\omega_{11}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{n}_1^1\\ \underset{\sim}{n}_2^1\\ \underset{\sim}{n}_3^1\end{Bmatrix}+\begin{bmatrix}\ell_2\omega_{23} & 0 & -\ell_2\omega_{21}\end{bmatrix}\begin{Bmatrix}\underset{\sim}{n}_1^2\\ \underset{\sim}{n}_2^2\\ \underset{\sim}{n}_3^2\end{Bmatrix}\\
&=\begin{bmatrix}\ell_1\omega_{13} & 0 & -\ell_1\omega_{11}\end{bmatrix}\begin{bmatrix}R_1\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix}+\begin{bmatrix}\ell_2\omega_{23} & 0 & -\ell_2\omega_{21}\end{bmatrix}\begin{bmatrix}R_2\end{bmatrix}\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix}
\end{aligned}
$$

So, the components of ${}^R\underset{\sim}{v}_B$ in the base system are

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^T = \Big[\begin{bmatrix} \ell_1\omega_{13} & 0 & -\ell_1\omega_{11} \end{bmatrix}\begin{bmatrix} R_1 \end{bmatrix} + \begin{bmatrix} \ell_2\omega_{23} & 0 & -\ell_2\omega_{21} \end{bmatrix}\begin{bmatrix} R_2 \end{bmatrix}\Big]^T$$

$$= \Big[\begin{bmatrix} \ell_1\omega_{13} & 0 & -\ell_1\omega_{11} \end{bmatrix}\begin{bmatrix} R_1 \end{bmatrix}\Big]^T + \Big[\begin{bmatrix} \ell_2\omega_{23} & 0 & -\ell_2\omega_{21} \end{bmatrix}\begin{bmatrix} R_2 \end{bmatrix}\Big]^T$$

$$\Rightarrow \boxed{\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{bmatrix} R_1 \end{bmatrix}^T \begin{Bmatrix} \ell_1\omega_{13} \\ 0 \\ -\ell_1\omega_{11} \end{Bmatrix} + \begin{bmatrix} R_2 \end{bmatrix}^T \begin{Bmatrix} \ell_2\omega_{23} \\ 0 \\ -\ell_2\omega_{21} \end{Bmatrix}}$$

The ***transformation matrices*** of the links and the ***body-fixed components*** of the ***angular velocity vectors*** are as given in Unit 5.

$$\boxed{\begin{bmatrix} R_i \end{bmatrix} = \begin{bmatrix} C_{i1}C_{i3} - S_{i1}C_{i2}S_{i3} & S_{i1}C_{i3} + C_{i1}C_{i2}S_{i3} & S_{i2}S_{i3} \\ -C_{i1}S_{i3} - S_{i1}C_{i2}C_{i3} & -S_{i1}S_{i3} + C_{i1}C_{i2}C_{i3} & S_{i2}C_{i3} \\ S_{i1}S_{i2} & -C_{i1}S_{i2} & C_{i2} \end{bmatrix}} \quad \boxed{\begin{aligned} \omega_{i1} &= \dot{\theta}_{i1}S_{i2}S_{i3} + \dot{\theta}_{i2}C_{i3} \\ \omega_{i2} &= \dot{\theta}_{i1}S_{i2}C_{i3} - \dot{\theta}_{i2}S_{i3} \\ \omega_{i3} &= \dot{\theta}_{i3} + \dot{\theta}_{i1}C_{i2} \end{aligned}} \quad (i = 1, 2)$$

The same results can also be found by using the concepts for points ***fixed*** on bodies as presented in Unit 3. Using the ***body-fixed*** components of the ***angular velocities*** of the links given in Unit 5, the ***velocity*** of B may be written as

$${}^R\underset{\sim}{v}_B = {}^R\underset{\sim}{v}_A + {}^R\underset{\sim}{v}_{B/A} = {}^R\underset{\sim}{v}_{A/O} + {}^R\underset{\sim}{v}_{B/A} = \underbrace{\left({}^R\underset{\sim}{\omega}_{L_1} \times \underset{\sim}{r}_{A/O}\right)}_{{}^R\underset{\sim}{v}_{A/O}} + \underbrace{\left({}^R\underset{\sim}{\omega}_{L_2} \times \underset{\sim}{r}_{B/A}\right)}_{{}^R\underset{\sim}{v}_{B/A}} = \begin{vmatrix} \underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1 \\ \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & -\ell_1 & 0 \end{vmatrix} + \begin{vmatrix} \underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2 \\ \omega_{21} & \omega_{22} & \omega_{23} \\ 0 & -\ell_2 & 0 \end{vmatrix}$$

$$= \ldots$$

This is clearly going to produce the ***same results*** as above.

b) Using the method of ***direct differentiation*** (along with the derivative rule) as presented in Unit 2, the acceleration ${}^R\underset{\sim}{a}_P$ can be found as follows

$${}^R\underset{\sim}{a}_B = \frac{{}^Rd}{dt}\left({}^R\underset{\sim}{v}_B\right) = \frac{{}^Rd}{dt}\Big(\left(\ell_1\omega_{13}\right)\underset{\sim}{n}_1^1 + (0)\underset{\sim}{n}_2^1 + \left(-\ell_1\omega_{11}\right)\underset{\sim}{n}_3^1\Big) + \frac{{}^Rd}{dt}\Big(\left(\ell_2\omega_{23}\right)\underset{\sim}{n}_1^2 + (0)\underset{\sim}{n}_2^2 + \left(-\ell_2\omega_{21}\right)\underset{\sim}{n}_3^2\Big)$$

$$= \frac{{}^{L_1}d}{dt}\Big(\left(\ell_1\omega_{13}\right)\underset{\sim}{n}_1^1 + (0)\underset{\sim}{n}_2^1 + \left(-\ell_1\omega_{11}\right)\underset{\sim}{n}_3^1\Big) + \Big({}^R\underset{\sim}{\omega}_{L_1} \times \Big(\left(\ell_1\omega_{13}\right)\underset{\sim}{n}_1^1 + (0)\underset{\sim}{n}_2^1 + \left(-\ell_1\omega_{11}\right)\underset{\sim}{n}_3^1\Big)\Big)$$

$$+ \frac{{}^{L_2}d}{dt}\Big(\left(\ell_2\omega_{23}\right)\underset{\sim}{n}_1^2 + (0)\underset{\sim}{n}_2^2 + \left(-\ell_2\omega_{21}\right)\underset{\sim}{n}_3^2\Big) + \Big({}^R\underset{\sim}{\omega}_{L_2} \times \Big(\left(\ell_2\omega_{23}\right)\underset{\sim}{n}_1^2 + (0)\underset{\sim}{n}_2^2 + \left(-\ell_2\omega_{21}\right)\underset{\sim}{n}_3^2\Big)\Big)$$

Expanding the cross products using determinants gives

$$
\begin{aligned}
{}^R\underset{\sim}{a}_B &= \left(\left(\ell_1\dot{\omega}_{13}\right)\underset{\sim}{n}_1^1+(0)\underset{\sim}{n}_2^1+\left(-\ell_1\dot{\omega}_{11}\right)\underset{\sim}{n}_3^1\right)+\left(\left(\ell_2\dot{\omega}_{23}\right)\underset{\sim}{n}_1^2+(0)\underset{\sim}{n}_2^2+\left(-\ell_2\dot{\omega}_{21}\right)\underset{\sim}{n}_3^2\right)\\
&\quad+\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \ell_1\omega_{13} & 0 & -\ell_1\omega_{11}\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2\\ \omega_{21} & \omega_{22} & \omega_{23}\\ \ell_2\omega_{23} & 0 & -\ell_2\omega_{21}\end{vmatrix}\\
&=\left(\ell_1\alpha_{13}-\ell_1\omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1+\left(\ell_1\omega_{13}^2+\ell_1\omega_{11}^2\right)\underset{\sim}{n}_2^1+\left(-\ell_1\alpha_{11}-\ell_1\omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\\
&\quad+\left(\ell_2\alpha_{23}-\ell_2\omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2+\left(\ell_2\omega_{23}^2+\ell_2\omega_{21}^2\right)\underset{\sim}{n}_2^2+\left(-\ell_2\alpha_{21}-\ell_2\omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2
\end{aligned}
$$

Part of this result is expressed in the L_1 frame and part is expressed in the L_2 frame. Using the transformation matrices for the links, the components in the base frame may be written as

$$
\begin{Bmatrix}A_1\\A_2\\A_3\end{Bmatrix}=\left[R_1\right]^T\begin{Bmatrix}\ell_1\alpha_{13}-\ell_1\omega_{11}\omega_{12}\\ \ell_1\omega_{13}^2+\ell_1\omega_{11}^2\\ -\ell_1\alpha_{11}-\ell_1\omega_{12}\omega_{13}\end{Bmatrix}+\left[R_2\right]^T\begin{Bmatrix}\ell_2\alpha_{23}-\ell_2\omega_{21}\omega_{22}\\ \ell_2\omega_{23}^2+\ell_2\omega_{21}^2\\ -\ell_2\alpha_{21}-\ell_2\omega_{22}\omega_{23}\end{Bmatrix}
$$

The ***transformation matrices*** and the ***body-fixed components*** of the ***angular velocity vectors*** are as defined in part (a), and the ***body-fixed components*** of the ***angular acceleration vectors*** are as given in Unit 5.

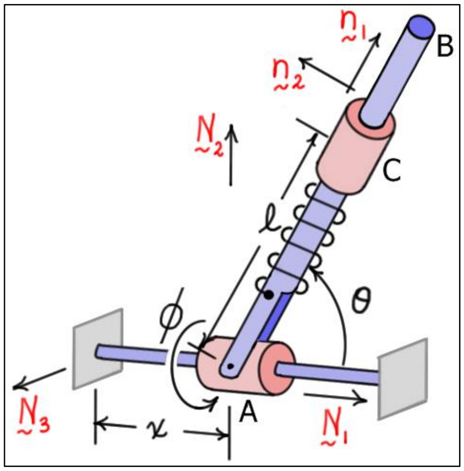
$$
\begin{aligned}
\alpha_{i1}&=\dot{\omega}_{i1}=\ddot{\theta}_{i1}S_{i2}S_{i3}+\dot{\theta}_{i1}\dot{\theta}_{i2}C_{i2}S_{i3}+\dot{\theta}_{i1}\dot{\theta}_{i3}S_{i2}C_{i3}+\ddot{\theta}_{i2}C_{i3}-\dot{\theta}_{i2}\dot{\theta}_{i3}S_{i3}\\
\alpha_{i2}&=\dot{\omega}_{i2}=\ddot{\theta}_{i1}S_{i2}C_{i3}+\dot{\theta}_{i1}\dot{\theta}_{i2}C_{i2}C_{i3}-\dot{\theta}_{i1}\dot{\theta}_{i3}S_{i2}S_{i3}-\ddot{\theta}_{i2}S_{i3}-\dot{\theta}_{i2}\dot{\theta}_{i3}C_{i3}\qquad (i=1,2)\\
\alpha_{i3}&=\dot{\omega}_{i3}=\ddot{\theta}_{i3}+\ddot{\theta}_{i1}C_{i2}-\dot{\theta}_{i1}\dot{\theta}_{i2}S_{i2}
\end{aligned}
$$

The same results can also be found by using the concepts for points ***fixed on bodies*** as presented in Unit 3. Using the ***body-fixed*** components of the ***angular velocities*** and ***angular accelerations*** of the links given in Unit 5, the acceleration of *B* may be written as

$$
\begin{aligned}
{}^R\underset{\sim}{a}_B &= {}^R\underset{\sim}{a}_A+{}^R\underset{\sim}{a}_{B/A}={}^R\underset{\sim}{a}_{A/O}+{}^R\underset{\sim}{a}_{B/A}\\
&=\left({}^R\underset{\sim}{\alpha}_{L_1}\times\underset{\sim}{r}_{A/O}\right)+\left({}^R\underset{\sim}{\omega}_{L_1}\times{}^R\underset{\sim}{v}_{A/O}\right)+\left({}^R\underset{\sim}{\alpha}_{L_2}\times\underset{\sim}{r}_{B/A}\right)+\left({}^R\underset{\sim}{\omega}_{L_2}\times{}^R\underset{\sim}{v}_{B/A}\right)\\
&=\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \alpha_{11} & \alpha_{12} & \alpha_{13}\\ 0 & -\ell_1 & 0\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^1 & \underset{\sim}{n}_2^1 & \underset{\sim}{n}_3^1\\ \omega_{11} & \omega_{12} & \omega_{13}\\ \ell_1\omega_{13} & 0 & -\ell_1\omega_{11}\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ 0 & -\ell_2 & 0\end{vmatrix}+\begin{vmatrix}\underset{\sim}{n}_1^2 & \underset{\sim}{n}_2^2 & \underset{\sim}{n}_3^2\\ \omega_{21} & \omega_{22} & \omega_{23}\\ \ell_2\omega_{23} & 0 & -\ell_2\omega_{21}\end{vmatrix}\\
&=\left(\ell_1\alpha_{13}-\ell_1\omega_{11}\omega_{12}\right)\underset{\sim}{n}_1^1+\left(\ell_1\omega_{13}^2+\ell_1\omega_{11}^2\right)\underset{\sim}{n}_2^1+\left(-\ell_1\alpha_{11}-\ell_1\omega_{12}\omega_{13}\right)\underset{\sim}{n}_3^1\\
&\quad+\left(\ell_2\alpha_{23}-\ell_2\omega_{21}\omega_{22}\right)\underset{\sim}{n}_1^2+\left(\ell_2\omega_{23}^2+\ell_2\omega_{21}^2\right)\underset{\sim}{n}_2^2+\left(-\ell_2\alpha_{21}-\ell_2\omega_{22}\omega_{23}\right)\underset{\sim}{n}_3^2\\
&=\ldots
\end{aligned}
$$

These are clearly the ***same results*** as found above.

Example 3:



The system shown consists of three bodies – collar A, bar AB, and collar C. Collar A ***slides along*** and ***rotates about*** the fixed horizontal bar. Its displacement (x) and rotation (ϕ) are both in the ***fixed*** $\underset{\sim}{N}_1$ direction. This type of connection is often referred to as a ***cylindrical joint***. Bar AB is pinned to collar A. Its ***orientation*** relative to the collar (and the horizontal bar) is defined by the angle θ. Finally, collar C is assumed to simply ***translate*** along the bar AB and not rotate about it. Its position relative to A is given by the length ℓ, and its motion along the bar is restricted by the attached spring.

Given this setup, the orientation of the bar frame $AB:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ relative to the ground frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ may be described by a 1-3 ***body-fixed*** rotation sequence. To orient the bar relative to the ground, first align the two frames. In this position, the bar lies along the $\underset{\sim}{N}_1$ direction, and the pin axis at A is aligned with the $\underset{\sim}{N}_3$ direction. Then rotate the bar first through an angle ϕ about the $\underset{\sim}{N}_1$ direction (simulating the rotation of collar A), and then rotate the bar through an angle θ about the $\underset{\sim}{n}_3$ direction (simulating the pin rotation at A).

Find:

a) the components of ${}^R\underset{\sim}{v}_C$ the ***velocity*** of C in R in the ***ground*** frame

b) the components of ${}^R\underset{\sim}{a}_C$ the ***acceleration*** of C in R in the ***ground*** frame

Solution:

a) Using the method of ***direct differentiation*** (along with the derivative rule) as presented in Unit 2, the velocity ${}^R\underset{\sim}{v}_C$ can be found as follows

$$
\begin{aligned}
{}^R\underset{\sim}{v}_C &= \frac{{}^Rd}{dt}\left(\underset{\sim}{r}_C\right)=\frac{{}^Rd}{dt}\left(\underset{\sim}{r}_A+\underset{\sim}{r}_{C/A}\right)=\frac{{}^Rd}{dt}\left(x\,\underset{\sim}{N}_1+\ell\,\underset{\sim}{n}_1\right)=\frac{{}^Rd}{dt}\left(x\,\underset{\sim}{N}_1\right)+\frac{{}^Rd}{dt}\left(\ell\,\underset{\sim}{n}_1\right)\\
&= \dot{x}\,\underset{\sim}{N}_1+\frac{{}^{AB}d}{dt}\left(\ell\,\underset{\sim}{n}_1\right)+\left({}^R\underset{\sim}{\omega}_{AB}\times\ell\,\underset{\sim}{n}_1\right)=\dot{x}\,\underset{\sim}{N}_1+\dot{\ell}\,\underset{\sim}{n}_1+\begin{vmatrix}\underset{\sim}{n}_1 & \underset{\sim}{n}_2 & \underset{\sim}{n}_3\\ \omega_1 & \omega_2 & \omega_3\\ \ell & 0 & 0\end{vmatrix}\\
&= \dot{x}\,\underset{\sim}{N}_1+\dot{\ell}\,\underset{\sim}{n}_1+\left(\ell\omega_3\right)\underset{\sim}{n}_2+\left(-\ell\omega_2\right)\underset{\sim}{n}_3
\end{aligned}
$$

Using the transformation matrix for a 1-3 rotation sequence, the ground-frame components of ${}^R\underset{\sim}{v}_C$ may be written as

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ 0 \\ 0 \end{Bmatrix} + [R]^T \begin{Bmatrix} \dot{\ell} \\ \ell\,\omega_3 \\ -\ell\,\omega_2 \end{Bmatrix}$$

The ***transformation matrix*** and the ***body-fixed components*** of the ***angular velocity*** vector may be found using the techniques presented in Unit 5. First, align the unit vectors $\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3 \right)$ with the ground-frame unit vectors $\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3 \right)$. Then, rotate about an angle ϕ about the $\underset{\sim}{N}_1 = \underset{\sim}{N}'_1$ direction, and then rotate about a second angle θ about the rotated $\underset{\sim}{N}'_3 = \underset{\sim}{n}_3$ direction. Using this procedure, the three sets of unit vectors for the orientations of bar *AB* may be related as follows.

$$\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = \left[R_1 \right]^T \begin{Bmatrix} \underset{\sim}{N}'_1 \\ \underset{\sim}{N}'_2 \\ \underset{\sim}{N}'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix}^T \begin{Bmatrix} \underset{\sim}{N}'_1 \\ \underset{\sim}{N}'_2 \\ \underset{\sim}{N}'_3 \end{Bmatrix} \qquad \begin{Bmatrix} \underset{\sim}{N}'_1 \\ \underset{\sim}{N}'_2 \\ \underset{\sim}{N}'_3 \end{Bmatrix} = \left[R_2 \right]^T \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix}$$

Recall from Unit 5 that C_ϕ and C_θ represent the cosines of angles ϕ and θ, and S_ϕ and S_θ represent the sines of angles ϕ and θ. The transformation matrix for bar *AB* is then found to be

$$[R] = \left[R_2 \right]\left[R_1 \right] = \begin{bmatrix} C_\theta & S_\theta & 0 \\ -S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi & C_\phi \end{bmatrix} = \begin{bmatrix} C_\theta & C_\phi S_\theta & S_\phi S_\theta \\ -S_\theta & C_\phi C_\theta & S_\phi C_\theta \\ 0 & -S_\phi & C_\phi \end{bmatrix}$$

As used above, the matrix $[R]^T$ transforms components of vector expressed in *AB* into components in the ground frame.

As in Unit 5, the ***body-fixed*** components of the ***angular velocity vectors*** of the links are found using the ***summation rule*** for angular velocities.

$${}^{R}\underset{\sim}{\omega}_{AB} = \dot{\phi}\,\underset{\sim}{N}'_1 + \dot{\theta}\,\underset{\sim}{n}_3 = \dot{\phi}\left(C_\theta\,\underset{\sim}{n}_1 - S_\theta\,\underset{\sim}{n}_2 \right) + \dot{\theta}\,\underset{\sim}{n}_3 \quad \Rightarrow \quad \begin{array}{l} \omega_1 = \dot{\phi}\,C_\theta \\ \omega_2 = -\dot{\phi}\,S_\theta \\ \omega_3 = \dot{\theta} \end{array}$$

The same results can be found using the concepts for ***points moving on bodies*** as presented in Unit 4. Using the ***body-fixed components*** of the ***angular velocity*** of bar *AB*, the ***velocity*** of *C* may be written as

$${}^{R}\underset{\sim}{v}_C = {}^{R}\underset{\sim}{v}_{\hat{C}} + {}^{AB}\underset{\sim}{v}_C = {}^{R}\underset{\sim}{v}_A + {}^{R}\underset{\sim}{v}_{\hat{C}/A} + {}^{AB}\underset{\sim}{v}_C$$ (C moves on *AB*, $\hat{C}$ is fixed on *AB* and coincides with C)

where

$$ {}^R\underset{\sim}{v}_{\hat{C}} = {}^R\underset{\sim}{v}_A + {}^R\underset{\sim}{v}_{\hat{C}/A} = \dot{x}\,\underset{\sim}{N}_1 + \left({}^R\underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{\hat{C}/A}\right) = \dot{x}\,\underset{\sim}{N}_2 + \left(\omega_1\,\underset{\sim}{n}_1 + \omega_2\,\underset{\sim}{n}_2 + \omega_3\,\underset{\sim}{n}_3\right)\times\left(\ell\,\underset{\sim}{n}_1\right) $$

$$ = \dot{x}\,\underset{\sim}{N}_1 + \ell\left(\omega_3\,\underset{\sim}{n}_2 - \omega_2\,\underset{\sim}{n}_3\right) $$

$$ {}^{AB}\underset{\sim}{v}_C = \dot{\ell}\;\underset{\sim}{n}_1 $$ (C has ***rectilinear motion*** on AB)

Substituting into the boxed equation gives

$$ {}^R\underset{\sim}{v}_C = \dot{x}\,\underset{\sim}{N}_1 + \dot{\ell}\,\underset{\sim}{n}_1 + \left(\ell\omega_3\right)\underset{\sim}{n}_2 + \left(-\ell\omega_2\right)\underset{\sim}{n}_3 $$ … ***same results as above***

b) Using the method of ***direct differentiation*** (along with the derivative rule) as presented in Unit 2, the acceleration ${}^R\underset{\sim}{a}_C$ can be found as follows

$$ {}^R\underset{\sim}{a}_C = \frac{{}^Rd}{dt}\left(\dot{x}\,\underset{\sim}{N}_1 + \dot{\ell}\,\underset{\sim}{n}_1 + \ell\omega_3\,\underset{\sim}{n}_2 - \ell\omega_2\,\underset{\sim}{n}_3\right) = \frac{{}^Rd}{dt}\left(\dot{x}\,\underset{\sim}{N}_1\right) + \frac{{}^Rd}{dt}\left(\dot{\ell}\,\underset{\sim}{n}_1 + \ell\omega_3\,\underset{\sim}{n}_2 - \ell\omega_2\,\underset{\sim}{n}_3\right) $$

$$ = \ddot{x}\,\underset{\sim}{N}_1 + \frac{{}^{AB}d}{dt}\left(\dot{\ell}\,\underset{\sim}{n}_1 + \ell\omega_3\,\underset{\sim}{n}_2 - \ell\omega_2\,\underset{\sim}{n}_3\right) + {}^R\underset{\sim}{\omega}_{AB} \times \left(\dot{\ell}\,\underset{\sim}{n}_1 + \ell\omega_3\,\underset{\sim}{n}_2 - \ell\omega_2\,\underset{\sim}{n}_3\right) $$

$$ = \ddot{x}\,\underset{\sim}{N}_1 + \left(\ddot{\ell}\,\underset{\sim}{n}_1 + \left(\dot{\ell}\omega_3 + \ell\dot{\omega}_3\right)\underset{\sim}{n}_2 - \left(\dot{\ell}\omega_2 + \ell\dot{\omega}_2\right)\underset{\sim}{n}_3\right) + \begin{vmatrix} \underset{\sim}{n}_1 & \underset{\sim}{n}_2 & \underset{\sim}{n}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \dot{\ell} & \ell\omega_3 & -\ell\omega_2 \end{vmatrix} $$

$$ = \ddot{x}\,\underset{\sim}{N}_1 + \left(\ddot{\ell} - \ell\omega_2^2 - \ell\omega_3^2\right)\underset{\sim}{n}_1 + \left(\dot{\ell}\omega_3 + \ell\dot{\omega}_3 + \dot{\ell}\omega_3 + \ell\omega_1\omega_2\right)\underset{\sim}{n}_2 + \left(-\dot{\ell}\omega_2 - \ell\dot{\omega}_2 + \ell\omega_1\omega_3 - \dot{\ell}\omega_2\right)\underset{\sim}{n}_3 $$

$$ = \ddot{x}\,\underset{\sim}{N}_1 + \left(\ddot{\ell} - \ell\omega_2^2 - \ell\omega_3^2\right)\underset{\sim}{n}_1 + \left(2\dot{\ell}\omega_3 + \ell\dot{\omega}_3 + \ell\omega_1\omega_2\right)\underset{\sim}{n}_2 + \left(-2\dot{\ell}\omega_2 - \ell\dot{\omega}_2 + \ell\omega_1\omega_3\right)\underset{\sim}{n}_3 $$

Using the transformation matrix defined above, the ground-frame components of ${}^R\underset{\sim}{a}_C$ are

$$ \boxed{\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} \ddot{x} \\ 0 \\ 0 \end{Bmatrix} + \left[R\right]^T \begin{Bmatrix} \ddot{\ell} - \ell\left(\omega_2^2 + \omega_3^2\right) \\ 2\dot{\ell}\omega_3 + \ell\left(\alpha_3 + \omega_1\omega_2\right) \\ \ell\left(\omega_1\omega_3 - \alpha_2\right) - 2\dot{\ell}\omega_2 \end{Bmatrix}} \quad \text{with} \quad \boxed{\begin{aligned} \alpha_1 &= \dot{\omega}_1 = \ddot{\phi}\,C_\theta - \dot{\phi}\,\dot{\theta}\,S_\theta \\ \alpha_2 &= \dot{\omega}_2 = -\ddot{\phi}\,S_\theta - \dot{\phi}\,\dot{\theta}\,C_\theta \\ \alpha_3 &= \dot{\omega}_3 = \ddot{\theta} \end{aligned}} $$

The same results may be found by using the concepts for ***points moving on bodies*** as presented in Unit 4. Using the ***body-fixed components*** of the ***angular velocity*** and ***angular acceleration*** of bar AB, the ***acceleration*** of C may be written as

$$ \boxed{{}^R\underset{\sim}{a}_C = {}^R\underset{\sim}{a}_{\hat{C}} + {}^{AB}\underset{\sim}{a}_C + 2\left({}^R\underset{\sim}{\omega}_{AB} \times {}^{AB}\underset{\sim}{v}_C\right)} $$ (C moves on AB, $\hat{C}$ is fixed on AB and coincides with C)

where, using the ***two-point formula***,

$$ {}^R\underset{\sim}{a}_{\hat{C}} = {}^R\underset{\sim}{a}_A + {}^R\underset{\sim}{a}_{\hat{C}/A} = \ddot{x}\,\underset{\sim}{N}_1 + \left({}^R\underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{C/A}\right) + \left({}^R\underset{\sim}{\omega}_{AB} \times {}^R\underset{\sim}{v}_{\hat{C}/A}\right) $$

Expanding each of the terms gives

$$
{}^{R}\underset{\sim}{a}_{\hat{C}} = \ddot{x}\,\underset{\sim}{N}_1 + \left[\left(\alpha_1\,\underset{\sim}{n}_1 + \alpha_2\,\underset{\sim}{n}_2 + \alpha_3\,\underset{\sim}{n}_3\right)\times\left(\ell\,\underset{\sim}{n}_1\right)\right] + \begin{vmatrix} \underset{\sim}{n}_1 & \underset{\sim}{n}_2 & \underset{\sim}{n}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & \ell\,\omega_3 & -\ell\,\omega_2 \end{vmatrix}
$$

$$
= \ddot{x}\,\underset{\sim}{N}_1 + \left[\ell\left(\alpha_3\,\underset{\sim}{n}_2 - \alpha_2\,\underset{\sim}{n}_3\right)\right] + \left[-\ell\left(\omega_2^2 + \omega_3^2\right)\underset{\sim}{n}_1 + \ell\,\omega_1\omega_2\,\underset{\sim}{n}_2 + \ell\,\omega_1\omega_3\,\underset{\sim}{n}_3\right]
$$

$$
= \ddot{x}\,\underset{\sim}{N}_1 + \left[-\ell\left(\omega_2^2 + \omega_3^2\right)\underset{\sim}{n}_1 + \ell\left(\alpha_3 + \omega_1\omega_2\right)\underset{\sim}{n}_2 + \ell\left(\omega_1\omega_3 - \alpha_2\right)\underset{\sim}{n}_3\right]
$$

${}^{AB}\underset{\sim}{a}_C = \ddot{\ell}\,\underset{\sim}{n}_1$ (C has ***rectilinear motion*** on AB)

$2\left({}^{R}\underset{\sim}{\omega}_{AB} \times {}^{AB}\underset{\sim}{v}_C\right) = 2\left(\omega_1\,\underset{\sim}{n}_1 + \omega_2\,\underset{\sim}{n}_2 + \omega_3\,\underset{\sim}{n}_3\right)\times\left(\dot{\ell}\,\underset{\sim}{n}_1\right) = 2\,\dot{\ell}\left(\omega_3\,\underset{\sim}{n}_2 - \omega_2\,\underset{\sim}{n}_3\right)$ (***Coriolis*** acceleration)

Substituting these results into the boxed equation gives

$$
\boxed{{}^{R}\underset{\sim}{a}_C = \ddot{x}\,\underset{\sim}{N}_1 + \left[\left(\ddot{\ell} - \ell\left(\omega_2^2 + \omega_3^2\right)\right)\underset{\sim}{n}_1 + \left(2\,\dot{\ell}\,\omega_3 + \ell\left(\alpha_3 + \omega_1\omega_2\right)\right)\underset{\sim}{n}_2 + \left(\ell\left(\omega_1\omega_3 - \alpha_2\right) - 2\,\dot{\ell}\,\omega_2\right)\underset{\sim}{n}_3\right]}
$$

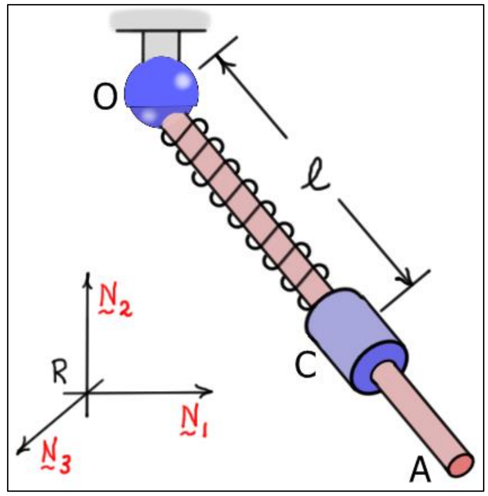
These are the ***same results*** as found above.

Notes:

1. In Example 2, the orientation of bar AB is described ***relative to the ground-frame*** rather than to the ***adjoining bar*** OA. In this case, this is easy to do because the interconnecting joint is spherical (ball-in-socket) – the ***analytical model*** of the connection is not limited by the choice of intermediate reference frames. If, however, the joint is a ***two-axis universal joint***, then the analytical model must ***accurately depict*** the directions about which rotation is allowed. In this case, one of these directions is ***fixed relative to the first body***, and the other is fixed ***relative to the second***. In cases like this, it may be more meaningful (or convenient) to describe the orientation of a body relative to the adjoining body rather than the ground. See Exercise 7.2.
2. If orientation angles are measured ***relative*** to a ***ground*** (inertial) frame, they are referred to as ***absolute*** angles. If they are measured ***relative*** to an ***adjoining body***, they are referred to as ***relative*** angles.
3. In Example 3, collar A is connected to the ground using a joint that allows ***translation*** and ***rotation*** of the collar ***about a single axis***. This type of joint is referred to as a ***cylindrical*** joint. As the joint connects bar AB to the ground, the two coordinates (x,ϕ) are ***absolute*** coordinates. If, however, this type of joint is used to connect two bodies within a system, it may be more ***natural*** to define the coordinates as measured ***relative*** to the adjoining body. In this way, they will be more conveniently used to described the ***relative*** motions of the two bodies.
4. In the above examples and in the Exercises below, the transformation matrices and angular velocity components are written in terms of ***orientation angles*** and their derivatives. In each case, however, they could have been written in terms of a set of four ***Euler parameters*** and their derivatives. See Unit 6.

Exercises:

7.1 The system shown consists of a bar *OA* and a collar *C*. Bar *OA* is connected to the ground using a ball-in-socket joint at *O*. As the bar swings, collar *C* translates along the bar. The variable length ℓ defines the position of *C* relative to *O*. The orientation of the bar relative to the ground-frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ is described by 3-1-3 body-fixed rotation sequence. When the orientation angles are all zero, the bar hangs vertically downward. Find a) the ground frame components of ${}^R\underset{\sim}{v}_C$ the velocity of collar *C* relative to the ground frame, and b) the ground-frame components of ${}^R\underset{\sim}{a}_C$ the acceleration of collar *C* relative to the ground frame.



Answers: (using ***body-fixed*** angular velocity components)

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = [R]^T \begin{Bmatrix} \ell\,\omega_3 \\ -\dot{\ell} \\ -\ell\,\omega_1 \end{Bmatrix}$$

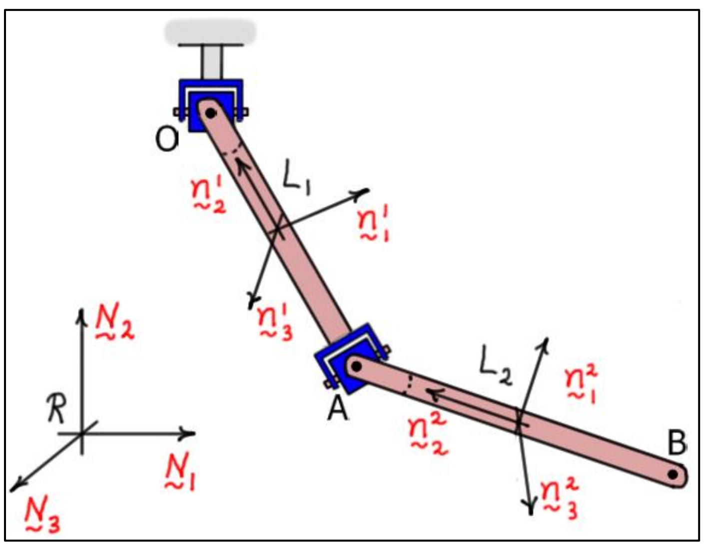
$$[R] = \begin{bmatrix} C_1C_3 - S_1C_2S_3 & S_1C_3 + C_1C_2S_3 & S_2S_3 \\ -C_1S_3 - S_1C_2C_3 & -S_1S_3 + C_1C_2C_3 & S_2C_3 \\ S_1S_2 & -C_1S_2 & C_2 \end{bmatrix}$$

$$\begin{aligned} \omega_1 &= \dot{\theta}_1 S_2 S_3 + \dot{\theta}_2 C_3 \\ \omega_2 &= \dot{\theta}_1 S_2 C_3 - \dot{\theta}_2 S_3 \\ \omega_3 &= \dot{\theta}_3 + \dot{\theta}_1 C_2 \end{aligned}$$

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = [R]^T \begin{Bmatrix} 2\dot{\ell}\,\omega_3 + \ell\left(\alpha_3 - \omega_1\omega_2\right) \\ -\ddot{\ell} + \ell\left(\omega_1^2 + \omega_3^2\right) \\ -2\dot{\ell}\,\omega_1 - \ell\left(\alpha_1 + \omega_2\omega_3\right) \end{Bmatrix}$$

$$\begin{aligned} \alpha_1 &= \dot{\omega}_1 = \ddot{\theta}_1 S_2 S_3 + \dot{\theta}_1\dot{\theta}_2 C_2 S_3 + \dot{\theta}_1\dot{\theta}_3 S_2 C_3 + \ddot{\theta}_2 C_3 - \dot{\theta}_2\dot{\theta}_3 S_3 \\ \alpha_2 &= \dot{\omega}_2 = \ddot{\theta}_1 S_2 C_3 + \dot{\theta}_1\dot{\theta}_2 C_2 C_3 - \dot{\theta}_1\dot{\theta}_3 S_2 S_3 - \ddot{\theta}_2 S_3 - \dot{\theta}_2\dot{\theta}_3 C_3 \\ \alpha_3 &= \dot{\omega}_3 = \ddot{\theta}_3 + \ddot{\theta}_1 C_2 - \dot{\theta}_1\dot{\theta}_2 S_2 \end{aligned}$$

7.2 The system shown is a ***three-dimensional double pendulum*** or ***arm***. The first link is connected to ground and the second link is connected to the first with ***universal joints*** at *O* and *A*, respectively. The ground frame is $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$ and the link frames are $L_i:\left(\underset{\sim}{n}_1^i,\underset{\sim}{n}_2^i,\underset{\sim}{n}_3^i\right)$ $(i=1,2)$. The ***orientation*** of L_1 is defined ***relative*** to *R* and the orientation of L_2 is defined ***relative*** to L_1 each with a 1-3 ***body-fixed*** rotation sequence.



Link *OA* is oriented relative to the ground frame by first rotating through an angle θ_{11} about the $\underset{\sim}{N}_1$ direction, and then rotating about an angle θ_{12} about the $\underset{\sim}{n}_3^1$ direction. Link *AB* is oriented relative to link *OA* by rotating first through an angle θ_{21} about the $\underset{\sim}{n}_1^1$ direction, and then through an angle θ_{22} about the $\underset{\sim}{n}_3^2$ direction. The ***lengths*** of the links are ℓ_1 and ℓ_2. Find a) the components of ${}^R\underset{\sim}{v}_B$ the ***velocity*** of *B* in *R* in the ***ground*** frame, and b) the components of ${}^R\underset{\sim}{a}_B$ the ***acceleration*** of *B* in *R* in the ***ground*** frame.

Answers: (using ***body-fixed*** angular velocity components)

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \left[R_1\right]^T \begin{Bmatrix} \ell_1\omega_{13} \\ 0 \\ -\ell_1\omega_{11} \end{Bmatrix} + \left[R_1\right]^T \left[R_2\right]^T \begin{Bmatrix} \ell_2\omega_{23} \\ 0 \\ -\ell_2\omega_{21} \end{Bmatrix}$$

$$\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \left[R_1\right]^T \begin{Bmatrix} \ell_1\alpha_{13} - \ell_1\omega_{11}\omega_{12} \\ \ell_1\omega_{13}^2 + \ell_1\omega_{11}^2 \\ -\ell_1\alpha_{11} - \ell_1\omega_{12}\omega_{13} \end{Bmatrix} + \left[R_1\right]^T \left[R_2\right]^T \begin{Bmatrix} \ell_2\alpha_{23} - \ell_2\omega_{21}\omega_{22} \\ \ell_2\omega_{23}^2 + \ell_2\omega_{21}^2 \\ -\ell_2\alpha_{21} - \ell_2\omega_{22}\omega_{23} \end{Bmatrix}$$

$$\left[R_i\right] = \begin{bmatrix} C_{i2} & S_{i2} & 0 \\ -S_{i2} & C_{i2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{i1} & S_{i1} \\ 0 & -S_{i1} & C_{i1} \end{bmatrix} = \begin{bmatrix} C_{i2} & C_{i1}S_{i2} & S_{i1}S_{i2} \\ -S_{i2} & C_{i1}C_{i2} & S_{i1}C_{i2} \\ 0 & -S_{i1} & C_{i1} \end{bmatrix} \quad (i = 1, 2) \ \ldots \text{ one for each link}$$

$${}^R\underset{\sim}{\omega}_{L_1} = \omega_{11}\,\underset{\sim}{n}_1^1 + \omega_{12}\,\underset{\sim}{n}_2^1 + \omega_{13}\,\underset{\sim}{n}_3^1 = \dot{\theta}_{11}\left(C_{12}\,\underset{\sim}{n}_1^1 - S_{12}\,\underset{\sim}{n}_2^1\right) + \dot{\theta}_{12}\,\underset{\sim}{n}_3^1$$

$${}^R\underset{\sim}{\omega}_{L_2} = {}^R\underset{\sim}{\omega}_{L_1} + {}^{L_1}\underset{\sim}{\omega}_{L_2} = \omega_{21}\,\underset{\sim}{n}_1^2 + \omega_{22}\,\underset{\sim}{n}_2^2 + \omega_{23}\,\underset{\sim}{n}_3^2$$

$$\begin{Bmatrix} \omega_{21} \\ \omega_{22} \\ \omega_{23} \end{Bmatrix} = \left[R_2\right] \begin{Bmatrix} C_{12}\dot{\theta}_{11} \\ -S_{12}\dot{\theta}_{11} \\ \dot{\theta}_{12} \end{Bmatrix} + \begin{Bmatrix} C_{22}\dot{\theta}_{21} \\ -S_{22}\dot{\theta}_{21} \\ \dot{\theta}_{22} \end{Bmatrix}$$

$$\begin{aligned} \alpha_{11} &= \dot{\omega}_{11} = C_{12}\ddot{\theta}_{11} - S_{12}\dot{\theta}_{11}\dot{\theta}_{12} \\ \alpha_{12} &= \dot{\omega}_{12} = -S_{12}\ddot{\theta}_{11} - C_{12}\dot{\theta}_{11}\dot{\theta}_{12} \\ \alpha_{13} &= \dot{\omega}_{13} = \ddot{\theta}_{12} \end{aligned}$$

$$\begin{Bmatrix} \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \end{Bmatrix} = \begin{Bmatrix} \dot{\omega}_{21} \\ \dot{\omega}_{22} \\ \dot{\omega}_{23} \end{Bmatrix} = \left[R_2\right] \begin{Bmatrix} C_{12}\ddot{\theta}_{11} - S_{12}\dot{\theta}_{11}\dot{\theta}_{12} \\ -S_{12}\ddot{\theta}_{11} - C_{12}\dot{\theta}_{11}\dot{\theta}_{12} \\ \ddot{\theta}_{12} \end{Bmatrix} + \left[\dot{R}_2\right] \begin{Bmatrix} C_{12}\dot{\theta}_{11} \\ -S_{12}\dot{\theta}_{11} \\ \dot{\theta}_{12} \end{Bmatrix} + \begin{Bmatrix} C_{22}\ddot{\theta}_{21} - S_{22}\dot{\theta}_{21}\dot{\theta}_{22} \\ -S_{22}\ddot{\theta}_{21} - C_{22}\dot{\theta}_{21}\dot{\theta}_{22} \\ \ddot{\theta}_{22} \end{Bmatrix}$$

$$\left[\dot{R}_2\right] = \frac{d}{dt}\begin{bmatrix} C_{22} & C_{21}S_{22} & S_{21}S_{22} \\ -S_{22} & C_{21}C_{22} & S_{21}C_{22} \\ 0 & -S_{21} & C_{21} \end{bmatrix} = \begin{bmatrix} -S_{22}\dot{\theta}_{22} & -S_{21}S_{22}\dot{\theta}_{21} + C_{21}C_{22}\dot{\theta}_{22} & C_{21}S_{22}\dot{\theta}_{21} + S_{21}C_{22}\dot{\theta}_{22} \\ -C_{22}\dot{\theta}_{22} & -S_{21}C_{22}\dot{\theta}_{21} - C_{21}S_{22}\dot{\theta}_{22} & C_{21}C_{22}\dot{\theta}_{21} - S_{21}S_{22}\dot{\theta}_{22} \\ 0 & -C_{21}\dot{\theta}_{21} & -S_{21}\dot{\theta}_{21} \end{bmatrix}$$

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