An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 7

Application of Concepts to Systems with Complex Interconnecting Joints

Summary

The basic kinematical concepts for mechanical systems were presented in Units 1 through 4. The concepts were applied to systems in which the relative orientation changes between adjoining bodies were described by a single rotation - simple revolute joints. In Units 5 and 6 concepts were presented for describing the relative orientation changes between any two adjoining bodies by using a set of up to three orientation angles or a set of four Euler parameters. This unit applies the concepts from Units 1 through 6 to analyze the kinematics of systems with more complex connecting joints. The major differences in applying the kinematical concepts of Units 1 through 4 are in the calculations of orientation, angular velocity and angular acceleration.

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Example 1:

The orientation of an aircraft relative to reference frame \( R \) is defined by a 3-2-1 **body-fixed** rotation sequence \((\psi, \theta, \phi)\). The **velocity** of the aircraft is given in body-fixed directions as \( R_{VG} = u \, b_1 + v \, b_2 + w \, b_3 \).

The **position** of a point \( P \) on the aircraft relative to the mass center \( G \) is given as \( R_{PG} = p_1 \, b_1 + p_2 \, b_2 + p_3 \, b_3 \). Point \( P \) is assumed to be **fixed** on the aircraft.

Find:

a) the components of \( R_{VP} \) the **velocity** of \( P \) in \( R \) in both the **body-fixed** and **base** frames

b) the components of \( R_{AP} \) the **acceleration** of \( P \) in \( R \) in both the **body-fixed** and **base** frames

Solution:

a) Using the method of **direct differentiation** as presented in Unit 2, \( R_{VP} \) can be found as follows

\[
R_{VP} = \frac{d}{dt}(R_P) = \frac{d}{dt}(R_G + R_{PG}) = \frac{d}{dt}(R_G) + \frac{d}{dt}(R_{PG}) = R_{VG} + \frac{d}{dt}(R_{PG}) + \left( R \omega_b \times R_{PG} \right)
\]

\[
= \begin{vmatrix}
    b_1 & b_2 & b_3 \\
p_1 & p_2 & p_3
\end{vmatrix} 
+ \begin{vmatrix}
    b_1 & b_2 & b_3 \\
p_1 & p_2 & p_3
\end{vmatrix} 
\]

\[
= (u \, b_1 + v \, b_2 + w \, b_3) + \omega_1 \, b_1 + \omega_2 \, b_2 + \omega_3 \, b_3
\]

\[
R_{VP} = \begin{pmatrix} u + \omega_2 p_3 - \omega_3 p_2 \\
                    v + \omega_3 p_1 - \omega_1 p_3 \\
                    w + \omega_1 p_2 - \omega_2 p_1 \end{pmatrix} b_3
\]

Here (as presented in Unit 5): \( \omega_1 = \dot{\psi} - \dot{\phi} S_\theta \), \( \omega_2 = \dot{\theta} C_\phi + \dot{\phi} C_\phi S_\phi \), \( \omega_3 = -\dot{\theta} S_\phi + \dot{\phi} C_\phi C_\phi \).

The same results can also be found by using the concepts for points **fixed on bodies** as presented in Unit 3. Using the **body-fixed** components of the **angular velocity** of the aircraft developed in Unit 5, the velocity of \( P \) may be written as

\[
R_{VP} = R_{VG} + R_{VP/G} = R_{VG} + \left( R \omega_b \times R_{PG} \right) = (u \, b_1 + v \, b_2 + w \, b_3) + \begin{vmatrix}
    b_1 & b_2 & b_3 \\
p_1 & p_2 & p_3
\end{vmatrix} \]

This clearly produces the **same result** as above.
The components \( V_i \) of \( \mathbf{v}_P \) in the base frame \( R \) may be found from the body-fixed components \( v_i \) \((i=1,2,3)\) (as defined above) using the coordinate transformation matrix for a 3-2-1 orientation angle sequence derived in Unit 5.

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = [R]^T \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
C_\psi C_\phi & S_\psi C_\phi & -S_\phi \\
C_\psi S_\phi - S_\psi C_\phi & S_\psi S_\phi + C_\psi C_\phi & C_\phi S_\phi \\
C_\psi S_\phi + S_\psi C_\phi & S_\psi S_\phi - C_\psi C_\phi & C_\phi S_\phi
\end{bmatrix}^T \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
\]

b) Using the method of direct differentiation (along with the derivative rule) as presented in Unit 2, \( \mathbf{a}_P \) can be found as follows

\[
\frac{d}{dt} \mathbf{a}_P = \frac{d}{dt} \mathbf{a}_G + \frac{d}{dt} \mathbf{a}_{P/G} = \frac{d}{dt} \mathbf{v}_G + \left( \mathbf{\omega}_B \times \mathbf{v}_{P/G} \right) + \left( \mathbf{\omega}_B \times \mathbf{v}_P \right)
\]

Or

\[
\mathbf{a}_P = \mathbf{a}_G + \mathbf{a}_{P/G}
\]

where

\[
\begin{align*}
a_1 &= \dot{u} + \alpha_2 p_3 - \alpha_3 p_2 + \omega_2 (w + \alpha_1 p_2 - \omega_2 p_1) - \omega_3 (v + \omega_3 p_1 - \omega_1 p_3) \\
a_2 &= \dot{v} + \alpha_3 p_1 - \alpha_1 p_3 + \omega_3 (u + \omega_2 p_3 - \omega_3 p_2) - \omega_1 (w + \omega_1 p_2 - \omega_2 p_1) \\
a_3 &= \dot{w} + \alpha_1 p_2 - \alpha_2 p_1 + \omega_1 (v + \omega_3 p_1 - \omega_1 p_3) - \omega_2 (u + \omega_2 p_3 - \omega_3 p_2)
\end{align*}
\]

This result can also be found using the concepts for points fixed on bodies as presented in Unit 3. Using the body-fixed components of the angular velocity and angular acceleration vectors of the aircraft developed in Unit 5, the acceleration of \( P \) may be written as

\[
\mathbf{a}_P = \mathbf{a}_G + \mathbf{a}_{P/G} = \frac{d}{dt} \mathbf{v}_G + \left( \mathbf{\omega}_B \times \mathbf{v}_{P/G} \right) + \left( \mathbf{\omega}_B \times \mathbf{v}_P \right)
\]
Using the derivative rule for the first term gives

\[
\frac{d}{dt} ( R \mathbf{a}_P ) = \left( \frac{d}{dt} ( R \mathbf{a}_B ) \right) \times ( R \mathbf{r}_{BG} ) + \left( \frac{d}{dt} ( R \mathbf{r}_{P/G} ) \right) + \left( \frac{d}{dt} ( R \omega_B ) \times ( R \mathbf{r}_{P/G} ) \right) + \left( R \omega_B \times \left( \frac{d}{dt} ( R \mathbf{r}_{P/G} ) \right) \right)
\]

\[
= \left( \dot{u} b_1 + \dot{v} b_2 + \dot{w} b_3 \right) + \left| \begin{array}{ccc}
 b_1 & b_2 & b_3 \\
 \omega_1 & \omega_2 & \omega_3 \\
 u & v & w \\
\end{array} \right| \left| \begin{array}{ccc}
 b_1 & b_2 & b_3 \\
 \alpha_1 & \alpha_2 & \alpha_3 \\
 p_1 & p_2 & p_3 \\
\end{array} \right|
\]

\[
= \left( \omega_2 p_3 - \omega_3 p_2 \right) + \left( \omega_3 p_1 - \omega_1 p_3 \right) + \left( \omega_1 p_2 - \omega_2 p_1 \right)
\]

\[
\Rightarrow \frac{d}{dt} ( R \mathbf{a}_P ) = a_1 b_1 + a_2 b_2 + a_3 b_3
\]

Expanding the determinants and combining terms gives the same components \( a_i \) (i = 1, 2, 3) found above using direct differentiation. The angular velocity components \( \omega_i \) (i = 1, 2, 3) are as defined in part (a), and the angular acceleration components \( \alpha_i \) (i = 1, 2, 3) are as computed in Unit 5.

\[
\alpha_1 = \dot{\omega}_1 = \ddot{\psi} S_\phi - \dot{\phi} C_\phi \\
\alpha_2 = \dot{\omega}_2 = \ddot{\theta} C_\phi - \dot{\phi} S_\phi + \dot{\psi} C_\theta S_\phi - \psi \dot{\theta} S_\theta S_\phi + \dot{\psi} \dot{\phi} C_\theta C_\phi \\
\alpha_3 = \dot{\omega}_3 = -\ddot{\phi} S_\phi - \dot{\theta} \dot{\phi} C_\phi + \dot{\psi} C_\theta C_\phi - \dot{\psi} \dot{\theta} S_\theta C_\phi - \dot{\psi} \dot{\phi} C_\theta S_\phi \\
\]

The components \( ( A_i ) \) of \( R \mathbf{a}_P \) in the base frame \( R \) may be found from the body-fixed components \( ( a_i \) as defined above) using the coordinate transformation matrix.

\[
\begin{bmatrix}
 A_1 \\
 A_2 \\
 A_3 \\
\end{bmatrix} = [R]^T \begin{bmatrix}
 a_1 \\
 a_2 \\
 a_3 \\
\end{bmatrix} = \begin{bmatrix}
 C_\psi C_\theta & S_\psi C_\theta & -S_\theta \\
 C_\psi S_\theta S_\phi - S_\psi C_\phi & S_\psi S_\theta S_\phi + C_\psi C_\phi & C_\phi S_\theta \\
 C_\psi S_\theta C_\phi + S_\psi S_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi & C_\phi C_\theta \\
\end{bmatrix} \begin{bmatrix}
 v_1 \\
 v_2 \\
 v_3 \\
\end{bmatrix}
\]
Example 2:

The system shown is a three-dimensional double pendulum or arm. The first link is connected to ground and the second link is connected to the first with ball and socket joints at O and A. The ground frame is $R : (N_1, N_2, N_3)$ and the link frames are $L_i : (n^i_1, n^i_2, n^i_3)$ ($i = 1, 2$). The orientation of each link is defined relative to $R$ using a 3-1-3 body-fixed rotation sequence. The lengths of the links are $\ell_1$ and $\ell_2$.

Find:

a) the components of $^R\dot{v}_B$ the velocity of $B$ in $R$ in the base frame

b) the components of $^R\ddot{a}_B$ the acceleration of $B$ in $R$ in the base frame

Solution:

a) Using the method of direct differentiation (along with the derivative rule) as presented in Unit 2, the velocity $^R\dot{v}_P$ can be found as follows

\[
^R\dot{v}_B = \frac{^R\dot{d}}{dt}(R_B) = \frac{^R\dot{d}}{dt}(R_{A/O} + R_{B/A}) = \frac{^R\dot{d}}{dt}(-\ell_1 n^1_2) + \frac{^R\dot{d}}{dt}(-\ell_2 n^2_2)
\]

\[
= \frac{^L\dot{d}}{dt}(-\ell_1 n^1_2) + \left(r_1 \omega_{L_1} \times -\ell_1 n^1_2\right) + \frac{^L\dot{d}}{dt}(-\ell_2 n^2_2) + \left(r_1 \omega_{L_2} \times -\ell_2 n^2_2\right)
\]

\[
= \begin{bmatrix} n^1_1 & n^1_2 & n^1_3 \\ n^1_2 & n^2_2 & n^2_3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \ell_1 \omega_{L_1} & 0 & 0 \\ -\ell_1 \omega_{L_1} & \ell_2 \omega_{L_2} & 0 \\ 0 & -\ell_2 \omega_{L_2} & \ell_2 \omega_{L_2} \end{bmatrix}
\]

\[
= \left(\ell_1 \omega_{L_1}\right)n^1_1 + (0)n^1_2 + (-\ell_1 \omega_{L_1})n^1_3 + \left(\ell_2 \omega_{L_2}\right)n^2_2 + (0)n^2_2 + (-\ell_2 \omega_{L_2})n^2_3
\]

\[
= \begin{bmatrix} \ell_1 \omega_{L_1} & 0 & -\ell_1 \omega_{L_1} \end{bmatrix} \begin{bmatrix} n^1_1 \\ n^1_2 \\ n^1_3 \end{bmatrix} + \begin{bmatrix} \ell_2 \omega_{L_2} & 0 & -\ell_2 \omega_{L_2} \end{bmatrix} \begin{bmatrix} n^2_2 \\ n^2_2 \\ n^2_3 \end{bmatrix}
\]

\[
= \begin{bmatrix} \ell_1 \omega_{L_1} & 0 & -\ell_1 \omega_{L_1} \end{bmatrix} \begin{bmatrix} R_1 \\ N_2 \\ N_3 \end{bmatrix} + \begin{bmatrix} \ell_2 \omega_{L_2} & 0 & -\ell_2 \omega_{L_2} \end{bmatrix} \begin{bmatrix} R_2 \\ N_2 \\ N_3 \end{bmatrix}
\]
So, the components of $R_{vb}$ in the base system are

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} [\ell_1 \omega_{13}] & [\ell_1 \omega_{11}] & [R_1] \\ [\ell_2 \omega_{23}] & [\ell_2 \omega_{21}] & [R_2] \end{bmatrix}^T$$

$$= \begin{bmatrix} [\ell_1 \omega_{13}] & [\ell_1 \omega_{11}] \\ [\ell_2 \omega_{23}] & [\ell_2 \omega_{21}] \end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \ell_1 \omega_{13} \\ 0 \\ -\ell_1 \omega_{11} \end{bmatrix}^T + \begin{bmatrix} \ell_2 \omega_{23} \\ 0 \\ -\ell_2 \omega_{21} \end{bmatrix}^T$$

The transformation matrices of the links and the body-fixed components of the angular velocity vectors are as given in Unit 5.

$$\begin{bmatrix} R_i \end{bmatrix} = \begin{bmatrix} C_{i_1}C_{i_2}S_{i_3} - S_{i_1}C_{i_2}S_{i_3} & S_{i_1}C_{i_2} + C_{i_1}C_{i_2}S_{i_3} & S_{i_2}S_{i_3} \\ -C_{i_1}S_{i_2} - S_{i_1}C_{i_2}C_{i_3} & -S_{i_1}S_{i_2} + C_{i_1}C_{i_2}C_{i_3} & S_{i_2}C_{i_3} \\ S_{i_1}S_{i_2} - C_{i_1}S_{i_2}C_{i_3} & -C_{i_1}S_{i_2}C_{i_3} & C_{i_2} \end{bmatrix}$$

$$\omega_{ij} = \dot{\theta}_{i_1}S_{i_2}S_{i_3} + \dot{\theta}_{i_2}C_{i_3}$$

$$\omega_{ij} = \dot{\theta}_{i_1}S_{i_2}C_{i_3} - \dot{\theta}_{i_2}S_{i_3} \quad (i = 1, 2)$$

The same results can also be found by using the concepts for points fixed on bodies as presented in Unit 3. Using the body-fixed components of the angular velocities of the links given in Unit 5, the velocity of $B$ may be written as

$$R_{vb} = R_{va} + R_{vb/a} = R_{va/o} + R_{vb/a} = \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \end{bmatrix} = \begin{bmatrix} \omega_{11} \\ \omega_{12} \\ \omega_{13} \end{bmatrix} + \begin{bmatrix} \omega_{21} \\ \omega_{22} \\ \omega_{23} \end{bmatrix}$$

$$= \ldots$$

This is clearly going to produce the same results as above.

b) Using the method of direct differentiation (along with the derivative rule) as presented in Unit 2, the acceleration $\ddot{R}_{vb}$ can be found as follows

$$\ddot{R}_{vb} = \frac{d^2}{dt^2} \begin{bmatrix} R_{vb} \end{bmatrix} = \frac{d}{dt} \left( \begin{bmatrix} \ell_1 \omega_{13} \\ 0 \\ -\ell_1 \omega_{11} \end{bmatrix} \right) \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \end{bmatrix} = \begin{bmatrix} \ell_1 \omega_{13} & 0 & -\ell_1 \omega_{11} \end{bmatrix} \begin{bmatrix} n_1^1 & n_2^1 & n_3^1 \\ n_1^2 & n_2^2 & n_3^2 \end{bmatrix}$$

Expanding the cross products using determinants gives
Part of this result is expressed in the $L_1$ frame and part is expressed in the $L_2$ frame. Using the transformation matrices for the links, the components in the base frame may be written as:

\[
\begin{align*}
\{A_1\} &= \begin{bmatrix} l_1\alpha_{13} - l_1\alpha_{11}\omega_{12} \\ l_1\alpha_{13} + l_1\omega_{11}^2 \\ -l_1\alpha_{11} - l_1\alpha_{12}\omega_{13} \end{bmatrix} + \begin{bmatrix} l_2\alpha_{23} - l_2\omega_{21}\omega_{22} \\ l_2\omega_{23}^2 + l_2\omega_{21}^2 \\ -l_2\alpha_{21} - l_2\omega_{22}\omega_{23} \end{bmatrix} \\
\{A_2\} &= \begin{bmatrix} A_1 \end{bmatrix}^T \\
\{A_3\} &= \begin{bmatrix} A_2 \end{bmatrix}^T 
\end{align*}
\]

The transformation matrices and the body-fixed components of the angular velocity vectors are as defined in part (a), and the body-fixed components of the angular acceleration vectors are as given in Unit 5.

\[
\begin{align*}
\alpha_{1i} &= \dot{\omega}_{1i} = \dot{\theta}_{i1}S_{i1} + \dot{\theta}_{i2}C_{i1}S_{i1} + \dot{\theta}_{i3}C_{i1}S_{i3} - \dot{\theta}_{i2}C_{i3}S_{i1} \\
\alpha_{2i} &= \dot{\omega}_{i2} = \dot{\theta}_{i1}S_{i2}C_{i1}S_{i3} + \dot{\theta}_{i2}C_{i2}C_{i3} - \dot{\theta}_{i2}S_{i3} - \dot{\theta}_{i3}C_{i1}S_{i3} \\
\alpha_{3i} &= \dot{\omega}_{i3} = \dot{\theta}_{i3} + \dot{\theta}_{i2}C_{i1} - \dot{\theta}_{i1}\theta_{i2}C_{i1} \\
\end{align*}
\]

The same results can also be found by using the concepts for points fixed on bodies as presented in Unit 3. Using the body-fixed components of the angular velocities and angular accelerations of the links given in Unit 5, the acceleration of $B$ may be written as:

\[
\begin{align*}
^R a_B &= ^R a_A + ^R a_{B/A} = ^R a_{A/O} + ^R a_{B/A} \\
&= \left( ^R a_{L_1} \times ^R \omega_{A/O} \right) + \left( ^R a_{L_1} \times ^R \omega_{A/O} \right) + \left( ^R a_{L_2} \times ^R \omega_{B/A} \right) + \left( ^R a_{L_2} \times ^R \omega_{B/A} \right) \\
&= \begin{bmatrix} n_1^1 & n_1^2 & n_1^3 \\ n_2^1 & n_2^2 & n_2^3 \\ n_3^1 & n_3^2 & n_3^3 \\ \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \end{bmatrix} + \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \\ \end{bmatrix} \\
&= \begin{bmatrix} l_1\alpha_{13} - l_1\omega_{11}\omega_{12} \\ l_1\omega_{11}^2 \\ -l_1\omega_{11} \end{bmatrix} + \begin{bmatrix} l_2\alpha_{23} - l_2\omega_{21}\omega_{22} \\ l_2\omega_{23}^2 + l_2\omega_{21}^2 \\ -l_2\omega_{21} \end{bmatrix} \\
&= \left( l_1\alpha_{13} - l_1\omega_{11}\omega_{12} \right) n_1^1 + \left( l_1\omega_{11}^2 \right) n_1^2 + \left( -l_1\omega_{11} \right) n_1^3 \\
&\quad + \left( l_2\alpha_{23} - l_2\omega_{21}\omega_{22} \right) n_2^1 + \left( l_2\omega_{23}^2 + l_2\omega_{21}^2 \right) n_2^2 + \left( -l_2\omega_{21} \right) n_2^3 \\
&= \ldots
\end{align*}
\]

These are clearly the same results as found above.
Example 3:

The system shown consists of three bodies – collar A, bar \( AB \), and collar \( C \). Collar A slides along and rotates about the fixed horizontal bar. Its displacement \((x)\) and rotation \((\phi)\) are both in the fixed \( N_1 \) direction. This type of connection is often referred to as a cylindrical joint. Bar \( AB \) is pinned to collar A. Its orientation relative to the collar (and the horizontal bar) is defined by the angle \( \theta \). Finally, collar \( C \) is assumed to simply translate along the bar \( AB \) and not rotate about it. Its position relative to \( A \) is given by the length \( \ell \), and its motion along the bar is restricted by the attached spring.

Given this setup, the orientation of the bar frame \( AB: (n_1, n_2, n_3) \) relative to the ground frame \( R: (N_1, N_2, N_3) \) may be described by a 1-3 body-fixed rotation sequence. To orient the bar relative to the ground, first align the two frames. In this position, the bar lies along the \( N_1 \) direction, and the pin axis at \( A \) is aligned with the \( N_3 \) direction. Then rotate the bar first through an angle \( \phi \) about the \( N_1 \) direction (simulating the rotation of collar \( A \)), and then rotate the bar through an angle \( \theta \) about the \( n_3 \) direction (simulating the pin rotation at \( A \)).

Find:

a) the components of \( R^R_C v \) the velocity of \( C \) in \( R \) in the ground frame
b) the components of \( R^R_C a \) the acceleration of \( C \) in \( R \) in the ground frame

Solution:

a) Using the method of direct differentiation (along with the derivative rule) as presented in Unit 2, the velocity \( R^R_C v \) can be found as follows

\[
R^R_C v = \frac{d}{dt} (x_C, n_3) = \frac{d}{dt} (x_A + \ell n_1) = \frac{d}{dt} (x N_1 + \ell n_1) = \frac{d}{dt} (x N_1) + \frac{d}{dt} (\ell n_1)
\]

\[
= \dot{x} N_1 + \frac{AB}{dt} (\ell n_1) + (R \omega_{AB} \times \ell n_1) = \dot{x} N_1 + \hat{\ell} n_1 + \begin{bmatrix} n_1 & n_2 & n_3 \\ n_1 \omega_1 & n_2 \omega_2 & n_3 \omega_3 \\ \ell & 0 & 0 \end{bmatrix}
\]

\[
= \dot{x} N_1 + \hat{\ell} n_1 + (\ell \omega_3) n_2 + (-\ell \omega_2) n_3
\]

Using the transformation matrix for a 1-3 rotation sequence, the ground-frame components of \( R^R_C v \) may be written as
The transformation matrix and the body-fixed components of the angular velocity vector may be found using the techniques presented in Unit 5. First, align the unit vectors \( \mathbf{u}_n \) with the ground-frame unit vectors \( \mathbf{u}_N \). Then, rotate about an angle \( \phi \) about the \( \mathbf{u}_N \) direction, and then rotate about a second angle \( \theta \) about the rotated \( \mathbf{u}_N \) direction. Using this procedure, the three sets of unit vectors for the orientations of bar \( AB \) may be related as follows.

\[
\begin{bmatrix}
N_1' \\
N_2' \\
N_3'
\end{bmatrix} =
\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix}
\]

Recall from Unit 5 that \( C_\phi \) and \( C_\theta \) represent the cosines of angles \( \phi \) and \( \theta \), and \( S_\phi \) and \( S_\theta \) represent the sines of angles \( \phi \) and \( \theta \). The transformation matrix for bar \( AB \) is then found to be

\[
\begin{bmatrix}
N_1' \\
N_2' \\
N_3'
\end{bmatrix} =
\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix}
\]

As used above, the matrix \( [R]^T \) transforms components of vector expressed in \( AB \) into components in the ground frame.

As in Unit 5, the body-fixed components of the angular velocity vector of the links are found using the summation rule for angular velocities.

\[
\mathbf{\omega}_{AB} = \dot{\phi} N_1' + \dot{\theta} N_3' = \dot{\phi} \left( C_\theta n_1 - S_\theta n_2 \right) + \dot{\theta} n_3
\]

The same results can be found using the concepts for points moving on bodies as presented in Unit 4. Using the body-fixed components of the angular velocity of bar \( AB \), the velocity of \( C \) may be written as

\[
\mathbf{v}_C = \mathbf{v}_C + \mathbf{v}_{C/AB} = \mathbf{v}_A + \mathbf{v}_{C/AB} + \mathbf{v}_{C/AB}
\]

where
\[
\dot{v}_C = R_{v_A} + R_{v_{\dot{C}A}} = \dot{x} N_1 + \left( R_{\omega_{AB} \times \Omega_{\dot{C}A}} \right) = \dot{x} N_2 + \left( \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \right) \times \left( \ell n_1 \right)
\]
\[
= \dot{x} N_1 + \ell \left( \omega_3 n_2 - \omega_2 n_3 \right)
\]
\[
\dot{v}_C = \ell n_1 \quad (C \text{ has rectilinear motion on } AB)
\]
Substituting into the boxed equation gives
\[
\dot{v}_C = \dot{x} N_1 + \ell n_1 + \left( \ell \omega_3 \right) n_2 + \left( -\ell \omega_2 \right) n_3 \quad \text{... same results as above}
\]

b) Using the method of direct differentiation (along with the derivative rule) as presented in Unit 2, the acceleration \( \dot{a}_C \) can be found as follows

\[
\dot{a}_C = R_{d} \left( \dot{x} N_1 + \ell n_1 + \ell \omega_3 n_2 - \ell \omega_2 n_3 \right) = R_{d} \left( \dot{x} N_1 \right) + R_{d} \left( \ell n_1 + \ell \omega_3 n_2 - \ell \omega_2 n_3 \right)
\]
\[
= \dot{x} N_1 + \left( \ell n_1 + \ell \omega_3 n_2 - \ell \omega_2 n_3 \right) + R_{\omega_{AB} \times \left( \ell n_1 + \ell \omega_3 n_2 - \ell \omega_2 n_3 \right)}
\]
\[
= \dot{x} N_1 + \left( \ell n_1 + \ell \omega_3 + \ell \omega_3 \right) n_2 - \left( \ell \omega_2 + \ell \omega_2 \right) n_3 + \left| \begin{array}{ccc}
\ell & \ell \omega_3 & -\ell \omega_2 \\
\ell \omega_3 & \ell \omega_3 & -\ell \omega_2 \\
\ell \omega_2 & \ell \omega_2 & -\ell \omega_2
\end{array} \right|
\]
\[
= \dot{x} N_1 + \left( \ell - \ell \omega_3^2 - \ell \omega_2 \right) n_1 + \left( \ell \omega_3 + \ell \omega_3 + \ell \omega_3 + \ell \omega_3 \right) n_2 + \left( -\ell \omega_2 - \ell \omega_2 + \ell \omega_3 - \ell \omega_2 \right) n_3
\]

Using the transformation matrix defined above, the ground-frame components of \( \dot{a}_C \) are

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} = \begin{bmatrix}
\ddot{x} \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
R
\end{bmatrix}^T \begin{bmatrix}
\ell - \ell \left( \omega_3^2 + \omega_2 \right) \\
2 \ell \omega_3 + \ell \left( \alpha_3 + \omega_1 \omega_2 \right) \\
\ell \left( \omega_1 \omega_3 - \alpha_2 \right) - 2 \ell \omega_2
\end{bmatrix}
\]
\[
\text{with} \quad \begin{array}{l}
\alpha_1 = \omega_1 = \dot{\phi} C_\theta - \dot{\theta} S_\theta \\
\alpha_2 = \omega_2 = -\dot{\phi} S_\theta - \dot{\theta} C_\theta \\
\alpha_3 = \omega_3 = \ddot{\theta}
\end{array}
\]

The same results may be found by using the concepts for points moving on bodies as presented in Unit 4.

Using the body-fixed components of the angular velocity and angular acceleration of bar \( AB \), the acceleration of \( C \) may be written as

\[
\dot{a}_C = \dot{a}_C + \dot{A} + 2 \left( \omega_{\dot{AB}} \times \dot{v}_C \right)
\]

where, using the two-point formula,

\[
\dot{a}_C = \dot{a}_A + \dot{a}_{\dot{C}A} = \dot{x} N_1 + \left( \omega_{\dot{AB}} \times \Omega_{\dot{C}A} \right) + \left( \omega_{\dot{AB}} \times \dot{v}_C \right)
\]

Expanding each of the terms gives
\[
\begin{align*}
\mathbf{Rq_c} &= \dot{x}N_1 + \left[ (\alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3) \times (\ell n_1) \right] + \\
&= \dot{x}N_1 + \left[ \ell \left( \alpha_3 n_2 - \alpha_2 n_3 \right) \right] + \left[ -\ell \left( \omega_2^2 + \omega_3^2 \right) n_1 + \ell \omega_1 \omega_2 n_2 + \ell \omega_1 \omega_3 n_3 \right] \\
&= \dot{x}N_1 + \left[ -\ell \left( \omega_2^2 + \omega_3^2 \right) n_1 + \ell \left( \alpha_3 + \omega_1 \omega_2 \right) n_2 + \ell \left( \omega_1 \omega_3 - \alpha_2 \right) n_3 \right]
\end{align*}
\]

\[
\mathbf{a}^{AB}_C = \dot{\ell} n_1 \quad (C \text{ has rectilinear motion on } AB)
\]

Substituting these results into the boxed equation gives

\[
\mathbf{Rq_c} = \dot{x}N_1 + \left[ \left( \ell - \ell \left( \omega_2^2 + \omega_3^2 \right) \right) n_1 + 2 \ell \left( \omega_3 + \ell \left( \alpha_3 + \omega_1 \omega_2 \right) \right) n_2 + \ell \left( \omega_1 \omega_3 - \alpha_2 \right) - 2 \ell \omega_2 \right] n_3
\]

These are the same results as found above.

Notes:

1. In Example 2, the orientation of bar AB is described relative to the ground-frame rather than to the adjoining bar OA. In this case, this is easy to do because the interconnecting joint is spherical (ball-in-socket) – the analytical model of the connection is not limited by the choice of intermediate reference frames. If, however, the joint is a two-axis universal joint, then the analytical model must accurately depict the directions about which rotation is allowed. In this case, one of these directions is fixed relative to the first body, and the other is fixed relative to the second. In cases like this, it may be more meaningful (or convenient) to describe the orientation of a body relative to the adjoining body rather than the ground. See Exercise 7.2.

2. If orientation angles are measured relative to a ground (inertial) frame, they are referred to as absolute angles. If they are measured relative to an adjoining body, they are referred to as relative angles.

3. In Example 3, collar A is connected to the ground using a joint that allows translation and rotation of the collar about a single axis. This type of joint is referred to as a cylindrical joint. As the joint connects bar AB to the ground, the two coordinates \((x, \phi)\) are absolute coordinates. If, however, this type of joint is used to connect two bodies within a system, it may be more natural to define the coordinates as measured relative to the adjoining body. In this way, they will be more conveniently used to described the relative motions of the two bodies.

4. In the above examples and in the Exercises below, the transformation matrices and angular velocity components are written in terms of orientation angles and their derivatives. In each case, however, they could have been written in terms of a set of four Euler parameters and their derivatives. See Unit 6.
Exercises:

7.1 The system shown consists of a bar \( OA \) and a collar \( C \). Bar \( OA \) is connected to the ground using a ball-in-socket joint at \( O \). As the bar swings, collar \( C \) translates along the bar. The variable length \( \ell \) defines the position of \( C \) relative to \( O \). The orientation of the bar relative to the ground-frame \( R: (N_1, N_2, N_3) \) is described by 3-1-3 body-fixed rotation sequence. When the orientation angles are all zero, the bar hangs vertically downward. Find a) the ground frame components of \( R_{VC} \) the velocity of collar \( C \) relative to the ground frame, and b) the ground-frame components of \( R_{AC} \) the acceleration of collar \( C \) relative to the ground frame.

Answers: (using body-fixed angular velocity components)

\[
\begin{align*}
V_1 & = \begin{bmatrix} \ell \omega_3 \\ -\ell \omega_1 \\ -\ell \omega_2 \end{bmatrix} \\
V_2 & = \begin{bmatrix} C_1C_3 - S_1C_2S_3 \\ -C_1S_3 - S_1C_2C_3 \\ S_1S_2 \\ -S_1C_3 + C_1C_2S_3 \\ -S_1S_3 + C_1C_2C_3 \\ -C_1S_2 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\omega_1 & = \dot{\theta}_1S_2S_3 + \dot{\theta}_2C_3 \\
\omega_2 & = \dot{\theta}_1S_2C_3 - \dot{\theta}_2S_3 \\
\omega_3 & = \dot{\theta}_3 + \ddot{\theta}_1C_3
\end{align*}
\]

7.2 The system shown is a three-dimensional double pendulum or arm. The first link is connected to ground and the second link is connected to the first with universal joints at \( O \) and \( A \), respectively. The ground frame is \( R:(N_1, N_2, N_3) \) and the link frames are \( L_i: (n_1^i, n_2^i, n_3^i) \) \((i = 1, 2)\). The orientation of \( L_1 \) is defined relative to \( R \) and the orientation of \( L_2 \) is defined relative to \( L_1 \) each with a 1-3 body-fixed rotation sequence.
Link $OA$ is oriented relative to the ground frame by first rotating through an angle $\theta_{11}$ about the $N_1$ direction, and then rotating about an angle $\theta_{21}$ about the $n_1^1$ direction. Link $AB$ is oriented relative to link $OA$ by rotating first through an angle $\theta_{21}$ about the $n_1^1$ direction, and then through an angle $\theta_{22}$ about the $n_2^2$ direction. The lengths of the links are $\ell_1$ and $\ell_2$. Find a) the components of $R_{\omega_B}$ the velocity of $B$ in $R$ in the ground frame, and b) the components of $R_{\ddot{\omega_B}}$ the acceleration of $B$ in $R$ in the ground frame.

Answers: (using body-fixed angular velocity components)

$$
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix} = 
\begin{bmatrix}
R_1 \end{bmatrix}^T 
\begin{bmatrix}
\ell_1 \omega_{13} & 0 & -\ell_1 \omega_{11} \\
0 & \ell_1 \omega_{13} - \ell_1 \omega_{11} \omega_{12} & -\ell_1 \omega_{11} - \ell_1 \omega_{12} \omega_{13} \\
-\ell_2 \omega_{21} & -\ell_2 \omega_{22} & 0
\end{bmatrix} + 
\begin{bmatrix}
R_1 \\
R_2 \\
R_3
\end{bmatrix}^T 
\begin{bmatrix}
\ell_2 \omega_{23} \\
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} = 
\begin{bmatrix}
R_1 \end{bmatrix}^T 
\begin{bmatrix}
\ell_1 \alpha_{13} - \ell_1 \omega_{11} \omega_{12} & 0 & 0 \\
0 & \ell_1 \omega_{13} & 0 \\
-\ell_2 \omega_{21} & -\ell_2 \omega_{22} & 0
\end{bmatrix} + 
\begin{bmatrix}
R_1 \\
R_2 \\
R_3
\end{bmatrix}^T 
\begin{bmatrix}
\ell_2 \alpha_{23} - \ell_2 \omega_{21} \omega_{22} \\
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
R_{\omega_L_1} \\
R_{\omega_L_2}
\end{bmatrix} = \omega_{11} n_1^1 + \omega_{12} n_2^1 + \omega_{13} n_3^1 = \hat{\theta}_{11} (C_{12} n_1^1 - S_{12} n_2^1) + \hat{\theta}_{12} n_3^1
$$

$$
\begin{bmatrix}
R_{\ddot{\omega}_L_1} \\
R_{\ddot{\omega}_L_2}
\end{bmatrix} = \begin{bmatrix}
R_{21} \\
R_{22} \\
R_{23} 
\end{bmatrix} = 
\begin{bmatrix}
\omega_{21} \\
\omega_{22} \\
\omega_{23}
\end{bmatrix} = 
\begin{bmatrix}
C_{12} \hat{\theta}_{11} \\
-C_{12} \hat{\theta}_{11} \\
-C_{22} \hat{\theta}_{21}
\end{bmatrix} + 
\begin{bmatrix}
\dot{\theta}_{11} \\
\dot{\theta}_{12} \\
\dot{\theta}_{21}
\end{bmatrix}
$$

\begin{align*}
\alpha_{11} &= \dot{\omega}_{11} = C_{12} \ddot{\theta}_{11} - S_{12} \dot{\theta}_{11} \dot{\theta}_{12} \\
\alpha_{12} &= \dot{\omega}_{12} = -S_{12} \ddot{\theta}_{11} - C_{12} \dot{\theta}_{11} \dot{\theta}_{12} \\
\alpha_{13} &= \dot{\omega}_{13} = \ddot{\theta}_{12}
\end{align*}

\begin{align*}
\begin{bmatrix}
\alpha_{21} \\
\alpha_{22} \\
\alpha_{23}
\end{bmatrix} &= 
\begin{bmatrix}
\ddot{\omega}_{21} \\
\ddot{\omega}_{22} \\
\ddot{\omega}_{23}
\end{bmatrix} = 
\begin{bmatrix}
R_{21} \\
-R_{12} \ddot{\theta}_{11} - C_{12} \dot{\theta}_{11} \dot{\theta}_{12} \\
-R_{12} \ddot{\theta}_{11} + C_{12} \dot{\theta}_{11} \dot{\theta}_{12}
\end{bmatrix} + 
\begin{bmatrix}
R_{22} \\
R_{23} \\
R_{24}
\end{bmatrix} = 
\begin{bmatrix}
C_{12} \ddot{\theta}_{11} \\
-C_{12} \dot{\theta}_{11} \dot{\theta}_{12} \\
C_{12} \dot{\theta}_{11}
\end{bmatrix} + 
\begin{bmatrix}
\ddot{\theta}_{11} \\
\ddot{\theta}_{12} \\
\ddot{\theta}_{21}
\end{bmatrix}
\end{align*}

\begin{align*}
\begin{bmatrix}
\ddot{R}_{21} \\
\ddot{R}_{22} \\
\ddot{R}_{23}
\end{bmatrix} &= \frac{d}{dt} 
\begin{bmatrix}
C_{22} & C_{21} S_{22} & S_{21} S_{22} \\
-S_{21} C_{22} & C_{21} & S_{21} C_{22} \\
0 & -S_{21} & C_{21}
\end{bmatrix} = 
\begin{bmatrix}
-S_{21} \ddot{\theta}_{21} & -S_{21} \ddot{\theta}_{21} + C_{21 C_{22}} \ddot{\theta}_{22} & C_{21} \ddot{\theta}_{11} + S_{21} C_{22} \ddot{\theta}_{22} \\
-C_{22} \ddot{\theta}_{22} & -C_{22} \ddot{\theta}_{22} - C_{21} \ddot{\theta}_{21} - C_{21} \ddot{\theta}_{22} & -C_{21} \ddot{\theta}_{21} - S_{21} S_{22} \ddot{\theta}_{22} \\
0 & -C_{21} \ddot{\theta}_{21} & -S_{21} \ddot{\theta}_{21}
\end{bmatrix}
\end{align*}
References: