An Introduction to Three-Dimensional, Rigid Body Dynamics

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Volume I: Kinematics

Unit 8

Introduction to Systems with Closed Kinematic Chains

Summary

This unit provides an ***introduction*** to systems that have ***closed kinematic chains***. Previous units focused on how to ***constrain*** the motion between ***adjoining bodies*** that form ***open-tree*** or ***open-chain*** systems. In systems that have ***closed kinematic chains*** (or ***closed loops***), two points within a chain of rigidly interconnected bodies have ***specified*** or ***constrained motion*** (relative to the ground, for example). This unit will discuss how to ***include additional kinematic equations*** (constraints) that apply when a system has such chains. The coverage is ***not meant to be comprehensive*** but should give the reader a solid introduction to the ***additional complexities*** involved with connecting bodies in three dimensional systems.

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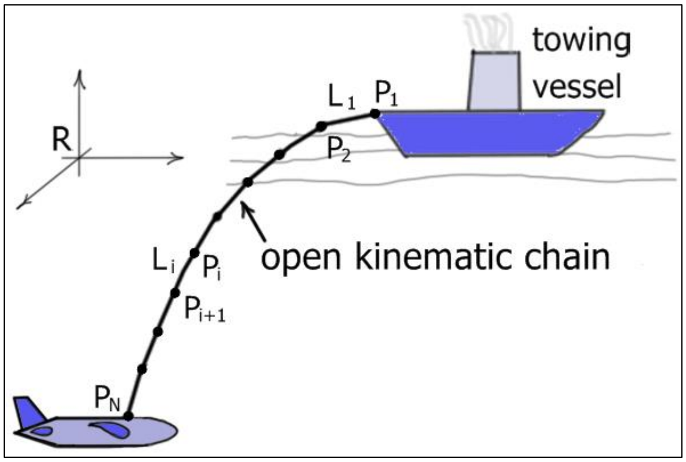
Systems with Closed Kinematic Chains (Constraints)



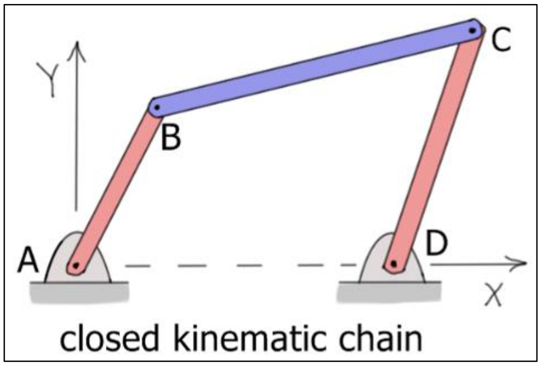
The systems discussed in previous units are all ***open tree*** or ***open chain*** systems. If a point within these systems has ***known motion*** relative to the ground-frame, then the motion of all other points in the system can be found by starting the kinematic analysis at that point. For example, consider a linked model of a ***towing cable*** shown in the diagram. The ***upper end*** of the cable (at point P_1) is attached to a ***towing vessel*** and could be considered to have specified (or known) motion.

Using the motion of P_1 as the starting point, the ***position***, ***velocity***, and ***acceleration*** of each of the points along the linked model of the towing cable can be written as

$$\underset{\sim}{r}_{P_{i+1}} = \underset{\sim}{r}_{P_1} + \sum_{j=1}^{i} \underset{\sim}{r}_{P_{j+1}/P_j} \qquad {}^R\underset{\sim}{v}_{P_{i+1}} = {}^R\underset{\sim}{v}_{P_1} + \sum_{j=1}^{i} {}^R\underset{\sim}{v}_{P_{j+1}/P_j} = {}^R\underset{\sim}{v}_{P_1} + \sum_{j=1}^{i}\left({}^R\underset{\sim}{\omega}_{L_j} \times \underset{\sim}{r}_{P_{j+1}/P_j} \right) \qquad (i = 1, \ldots, N-1)$$

$$\underset{\sim}{a}_{P_{i+1}} = {}^R\underset{\sim}{a}_{P_1} + \sum_{j=1}^{i} {}^R\underset{\sim}{a}_{P_{j+1}/P_j} = {}^R\underset{\sim}{a}_{P_1} + \sum_{j=1}^{i}\left({}^R\underset{\sim}{\alpha}_{L_i} \times \underset{\sim}{r}_{P_{j+1}/P_j} \right) + \sum_{j=1}^{i}\left({}^R\underset{\sim}{\omega}_{L_i} \times {}^R\underset{\sim}{v}_{P_{j+1}/P_j} \right) \qquad (i = 1, \ldots, N-1)$$

Given results for the motion of point P_N, kinematical equations of motion for the towed vehicle can be written as well. Since P_1 is the only point in the system with known motion, no further kinematical equations are required.



In contrast, many mechanical systems such as simple mechanisms have ***multiple points*** of ***known*** or ***constrained motion***. If these points are ***rigidly*** connected by bodies within the system, the ***connection*** is said to form a ***closed-loop*** or ***closed kinematic chain***. Closed kinematic chains place additional restrictions (or ***constraints***) on the motion of a system in addition to those of interconnecting joints.

Consider, for example, the four-bar mechanism shown in the diagram. Starting at point A, the ***position vector*** of point C may be written as

$$\boxed{\underset{\sim}{r}_C = \underset{\sim}{r}_{B/A} + \underset{\sim}{r}_{C/B} = \underset{\sim}{r}_{D/A} + \underset{\sim}{r}_{C/D}} \quad \Rightarrow \quad \boxed{\underset{\sim}{r}_{B/A} + \underset{\sim}{r}_{C/B} - \underset{\sim}{r}_{C/D} - \underset{\sim}{r}_{D/A} = 0}$$

The latter equation is referred to as ***loop-closure equation***, and it must be satisfied at all times for the mechanism to remain ***intact***.

The ***velocity*** of *C* may be found by ***differentiating*** its position vector or by using the kinematic formula for ***two points fixed on a rigid body***. Using the latter approach,

$$\boxed{\underset{\sim}{v}_C = \underset{\sim}{v}_B + \underset{\sim}{v}_{C/B} = \underbrace{\underset{\sim}{v}_A}_{\text{zero}} + \underset{\sim}{v}_{B/A} + \underset{\sim}{v}_{C/B} = \underset{\sim}{v}_{B/A} + \underset{\sim}{v}_{C/B} = \left(\underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A}\right) + \left(\underset{\sim}{\omega}_{BC} \times \underset{\sim}{r}_{C/B}\right)}$$

Also, starting at point *D* the velocity of point *C* may be written as

$$\boxed{\underset{\sim}{v}_C = \underset{\sim}{v}_D + \underset{\sim}{v}_{C/D} = \underset{\sim}{v}_{C/D} = \left(\underset{\sim}{\omega}_{CD} \times \underset{\sim}{r}_{C/D}\right)}$$

Again, for the mechanism to remain ***intact***, both expressions must be true.

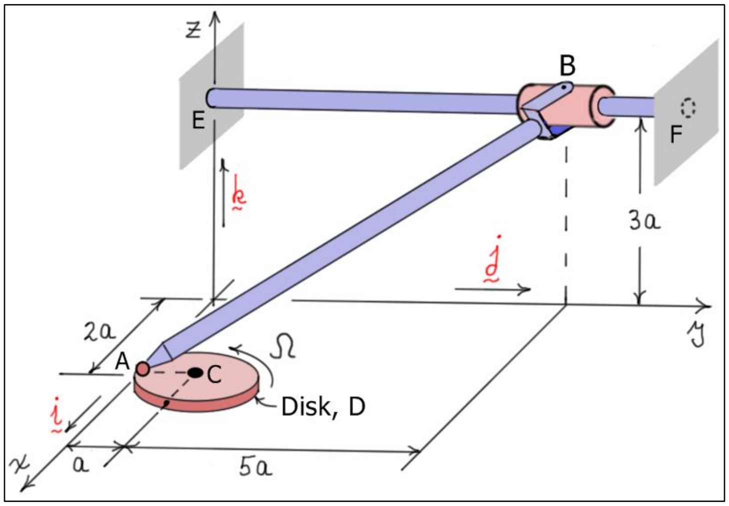
$$\boxed{\left(\underset{\sim}{\omega}_{CD} \times \underset{\sim}{r}_{C/D}\right) = \left(\underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A}\right) + \left(\underset{\sim}{\omega}_{BC} \times \underset{\sim}{r}_{C/B}\right)}$$

A ***similar expression*** can be written ***relating*** the ***accelerations*** of the three points.

Of the last three boxed equations of the velocity analysis, the first two ***inherently account*** for the nature of interconnecting joints. The third boxed equation, however, is an additional constraint associated with the closed kinematic chain. In a three-dimensional mechanism, ***additional*** constraint equations may be required to describe how the interconnecting joints ***restrict*** the motion.

Example 1: Slider-Crank Mechanism

The figure shows a three-dimensional slider-crank mechanism. Disk *D* rotates about its center *C* with angular velocity ${}^R\underset{\sim}{\omega}_D = \dot{\phi}\,\underset{\sim}{k} = \Omega\,\underset{\sim}{k}$. The center of the disk is in the *x*-*y* plane at the point $(2a,\ a,\ 0)$. Bar *AB* is attached to the disk with a ball and socket joint at *A* and is attached to the collar at *B* with a simple revolute joint. The collar can ***translate along*** and ***rotate about*** bar *EF* in the fixed *y* direction, and bar *AB* can ***rotate*** relative to the collar about an axis which is ***normal to*** the plane *AEB*. The ***size*** of the collar and the ***thickness*** of the disk are ***negligible***.



At the instant shown, $\phi = 0$, *A* has coordinates $(2a,\ 0,\ 0)$ and *B* has coordinates $(0,\ 6a,\ 3a)$.

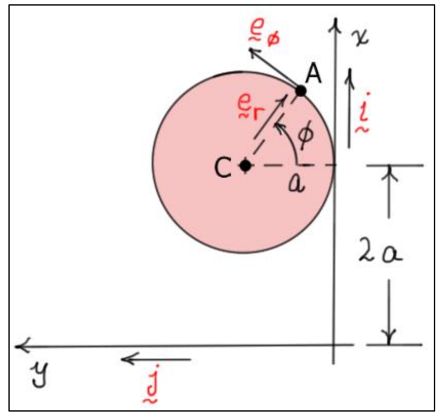
Find:

a) ${}^{R}\underset{\sim}{v}_{B}$ the velocity of B and ${}^{R}\underset{\sim}{\omega}_{AB}$ the angular velocity of bar AB as a function of disk angle ϕ

b) ${}^{R}\underset{\sim}{a}_{B}$ the acceleration of B and ${}^{R}\underset{\sim}{\alpha}_{AB}$ the angular acceleration of bar AB as a function of disk angle ϕ

Solution:

Geometric Preliminaries:



Position of A for a non-zero value of ϕ

From the diagram above with $\phi = 0$, the length of bar AB may be computed as follows:

$$\ell = \sqrt{\underset{\sim}{r}_{B/A} \cdot \underset{\sim}{r}_{B/A}} = \sqrt{\left(-2a\,\underset{\sim}{i} + 6a\,\underset{\sim}{j} + 3a\,\underset{\sim}{k}\right)^2} = \sqrt{(2a)^2 + (6a)^2 + (3a)^2} = \sqrt{49a^2} = 7a$$

When $\phi \neq 0$, the position vector of B relative to A may be written as

$$\underset{\sim}{r}_{B/A} = -\left(2a + aS_{\phi}\right)\underset{\sim}{i} + \left(y_B - a + aC_{\phi}\right)\underset{\sim}{j} + 3a\,\underset{\sim}{k}$$

As the system moves, B moves only in the y direction. Knowing the length of the bar is $\ell = 7a$, an expression for y_B the y-coordinate of collar B as a function of ϕ can be found as follows:

$$\ell^2 = 49a^2 = \left(2a + aS_{\phi}\right)^2 + \left(y_B - a + aC_{\phi}\right)^2 + (3a)^2$$

$$\Rightarrow \left(y_B - a + aC_{\phi}\right)^2 = 49a^2 - 9a^2 - \left(2a + aS_{\phi}\right)^2 = \left(40 - \left(2 + S_{\phi}\right)^2\right)a^2$$

$$\Rightarrow y_B - a\left(1 - C_{\phi}\right) = \left(\sqrt{40 - \left(2 + S_{\phi}\right)^2}\right)a$$

$$\Rightarrow \boxed{y_B = a\left[1 - C_{\phi} + \sqrt{40 - \left(2 + S_{\phi}\right)^2}\right] = a\,\hat{y}_B}$$

Kinematic Analysis:

a) As would be done with a ***two-dimensional*** slider-crank mechanism, the ***relative velocity equation*** may be applied to the end points of bar AB. That is,

$$\begin{aligned} {}^{R}\underset{\sim}{v}_{B} = v_B\,\underset{\sim}{j} &= {}^{R}\underset{\sim}{v}_{A} + {}^{R}\underset{\sim}{v}_{B/A} = a\Omega\,\underset{\sim}{e}_{\phi} + \left({}^{R}\underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A}\right) \\ &= a\Omega\left(C_{\phi}\,\underset{\sim}{i} + S_{\phi}\,\underset{\sim}{j}\right) + \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \omega_x & \omega_y & \omega_z \\ -a\left(2 + S_{\phi}\right) & y_B - a\left(1 - C_{\phi}\right) & 3a \end{vmatrix} \end{aligned}$$

Separating into a set of scalar equations and writing the result in matrix form gives

$$\begin{bmatrix} 0 & 0 & 3a & a(1-C_\phi)-y_B \\ 1 & 3a & 0 & a(2+S_\phi) \\ 0 & y_B-a(1-C_\phi) & a(2+S_\phi) & 0 \end{bmatrix} \begin{Bmatrix} v_B \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{Bmatrix} -a\Omega C_\phi \\ a\Omega S_\phi \\ 0 \end{Bmatrix}$$

This is a set of ***three linear algebraic equations*** with ***four unknowns*** v_B, ω_x, ω_y, and ω_z. To find a unique solution for ${}^R\underset{\sim}{\omega}_{AB}$ an ***additional constraint equation*** must be provided to indicate how the connecting joints at A and B ***restrict*** the motion of the bar. The ball and socket joint at A places no restriction on ${}^R\underset{\sim}{\omega}_{AB}$, but the joint at B allows only ***two rotational degrees of freedom*** (hence eliminating one rotational degree of freedom).

As mentioned in the problem statement, the bar can rotate with the collar about the *y*-direction, and it can rotate about the pin connecting it to the collar which is directed normal to the plane *AEB*. This eliminates any rotation about an axis which is normal (perpendicular) to both directions. The ***additional constraint equation*** associated with the connecting joint at *B* may be written as

$$ {}^R\underset{\sim}{\omega}_{AB} \cdot \underset{\sim}{u} = 0$$

where the vector $\underset{\sim}{u}$ must be normal to both the plane *AEB* and to the *y* direction.

$$\underset{\sim}{u} = \underbrace{\overbrace{\left(\underset{\sim}{r}_{A/E} \times \underset{\sim}{j}\right)}^{\text{normal to } AEB} \times \underset{\sim}{j}}_{\text{normal to both } AEB \text{ and } y} = \left[\left(a(2+S_\phi)\,\underset{\sim}{i} + a(1-C_\phi)\,\underset{\sim}{j} - 3a\,\underset{\sim}{k}\right) \times \underset{\sim}{j}\right] \times \underset{\sim}{j}$$

$$= \left[a(2+S_\phi)\,\underset{\sim}{k} + 3a\,\underset{\sim}{i}\right] \times \underset{\sim}{j} = -a(2+S_\phi)\,\underset{\sim}{i} + 3a\,\underset{\sim}{k}$$

Substituting for $\underset{\sim}{u}$ in the constraint equation and completing the dot product, gives the scalar equation

$$-a(2+S_\phi)\,\omega_x + 3a\,\omega_z = 0$$

Combining this equation with the ***first*** and ***third*** equations from above gives the following ***three simultaneous, linear algebraic equations*** for the ***three angular velocity components***. Note that the parameter "*a*" has been ***cancelled*** from each equation.

$$\begin{bmatrix} 0 & 3 & (1-C_\phi)-\hat{y}_B \\ \hat{y}_B-(1-C_\phi) & (2+S_\phi) & 0 \\ -(2+S_\phi) & 0 & 3 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [A]\{\omega\} = \begin{Bmatrix} -\Omega C_\phi \\ 0 \\ 0 \end{Bmatrix}$$

These equations may be solved to provide the ***angular velocity components*** of bar *AB* at any ***arbitrary*** disk angle ϕ. These components can then be used in the ***second*** of the kinematics equations to find the ***velocity*** of the collar. For example, if $\phi = 0$, then $\hat{y}_B = 6$ and the three equations reduce to

$$\begin{bmatrix} 0 & 3 & -6 \\ 6 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [A]\{\omega\} = \begin{Bmatrix} -\Omega \\ 0 \\ 0 \end{Bmatrix}$$

Solving this set of equations and using the results to find the scalar velocity v_B gives

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \frac{-1}{78} \begin{bmatrix} 6 & -9 & 12 \\ -18 & -12 & -36 \\ 4 & -6 & -18 \end{bmatrix} \begin{Bmatrix} -\Omega \\ 0 \\ 0 \end{Bmatrix} = \frac{\Omega}{39} \begin{Bmatrix} 3 \\ -9 \\ 2 \end{Bmatrix} \quad \text{and} \quad v_B = -3a\omega_x - 2a\omega_z = \frac{-a\Omega}{3} \quad (\text{at } \phi = 0)$$

b) Similarly, the ***relative acceleration equation*** may be applied to the end points of bar *AB* to find the angular acceleration of *AB* and the acceleration of the collar.

$${}^R\underset{\sim}{a}_B = a_B\,\underset{\sim}{j} = {}^R\underset{\sim}{a}_A + {}^R\underset{\sim}{a}_{B/A}$$

where

$$\begin{aligned} {}^R\underset{\sim}{a}_A &= -a\Omega^2\,\underset{\sim}{e}_r + a\dot{\Omega}\,\underset{\sim}{e}_\phi = -a\Omega^2\left(S_\phi\,\underset{\sim}{i} - C_\phi\,\underset{\sim}{j}\right) + a\dot{\Omega}\left(C_\phi\,\underset{\sim}{i} + S_\phi\,\underset{\sim}{j}\right) \\ &= \left(a\dot{\Omega}C_\phi - a\Omega^2 S_\phi\right)\underset{\sim}{i} + \left(a\dot{\Omega}S_\phi + a\Omega^2 C_\phi\right)\underset{\sim}{j} \end{aligned}$$

$$\begin{aligned} {}^R\underset{\sim}{a}_{B/A} &= \left({}^R\underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{B/A}\right) + {}^R\underset{\sim}{\omega}_{AB} \times \left({}^R\underset{\sim}{\omega}_{AB} \times \underset{\sim}{r}_{B/A}\right) \\ &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \alpha_x & \alpha_y & \alpha_z \\ -a\left(2+S_\phi\right) & y_B - a\left(1-C_\phi\right) & 3a \end{vmatrix} + \\ &\quad \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \omega_x & \omega_y & \omega_z \\ \left(3a\omega_y - \left(y_B - a\left(1-C_\phi\right)\right)\omega_z\right) & \left(-a\left(2+S_\phi\right)\omega_z - 3a\omega_x\right) & \left(\left(y_B - a\left(1-C_\phi\right)\right)\omega_x + a\left(2+S_\phi\right)\omega_y\right) \end{vmatrix} \end{aligned}$$

Expanding the cross-product terms gives

$$
\begin{aligned}
&{}^R\underset{\sim}{a}_{B/A} = \\
&\Big(3a\alpha_y - \big(y_B - a(1-C_\phi)\big)\alpha_z + \omega_y\Big(\big(y_B - a(1-C_\phi)\big)\omega_x + a(2+S_\phi)\omega_y\Big) + \omega_z\big(a(2+S_\phi)\omega_z + 3a\omega_x\big)\Big)\underset{\sim}{i} \\
&+\Big(-a(2+S_\phi)\alpha_z - 3a\alpha_x + \omega_z\Big(3a\omega_y - \big(y_B - a(1-C_\phi)\big)\omega_z\Big) - \omega_x\Big(\big(y_B - a(1-C_\phi)\big)\omega_x + a(2+S_\phi)\omega_y\Big)\Big)\underset{\sim}{j} \\
&+\Big(\big(y_B - a(1-C_\phi)\big)\alpha_x + a(2+S_\phi)\alpha_y + \omega_x\big(-a(2+S_\phi)\omega_z - 3a\omega_x\big) - \omega_y\Big(3a\omega_y - \big(y_B - a(1-C_\phi)\big)\omega_z\Big)\Big)\underset{\sim}{k}
\end{aligned}
$$

Separating into a set of scalar equations gives a set of ***three linear algebraic equations*** in ***four unknowns***.

$$
\begin{aligned}
&\begin{bmatrix} 0 & 0 & 3a & a(1-C_\phi) - y_B \\ 1 & 3a & 0 & a(2+S_\phi) \\ 0 & y_B - a(1-C_\phi) & a(2+S_\phi) & 0 \end{bmatrix}\begin{Bmatrix} a_B \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} \\
&\quad = \begin{Bmatrix} -a\dot{\Omega}C_\phi + a\Omega^2 S_\phi - \omega_y\Big(\big(y_B - a(1-C_\phi)\big)\omega_x + a(2+S_\phi)\omega_y\Big) - \omega_z\big(a(2+S_\phi)\omega_z + 3a\omega_x\big) \\ a\dot{\Omega}S_\phi + a\Omega^2 C_\phi + \omega_z\Big(3a\omega_y - \big(y_B - a(1-C_\phi)\big)\omega_z\Big) - \omega_x\Big(\big(y_B - a(1-C_\phi)\big)\omega_x + a(2+S_\phi)\omega_y\Big) \\ \omega_x\big(a(2+S_\phi)\omega_z + 3a\omega_x\big) + \omega_y\Big(3a\omega_y - \big(y_B - a(1-C_\phi)\big)\omega_z\Big) \end{Bmatrix}
\end{aligned}
$$

As before, an ***additional constraint equation*** is required. This equation may be found by ***differentiating*** the constraint from the velocity analysis. Using the product rule,

$$
\frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_{AB}\cdot\underset{\sim}{u}\right) = \left({}^R\underset{\sim}{\alpha}_{AB}\cdot\underset{\sim}{u}\right) + \left({}^R\underset{\sim}{\omega}_{AB}\cdot\frac{{}^R d\,\underset{\sim}{u}}{dt}\right) = 0
$$

Using the results derived above for the vector $\underset{\sim}{u}$ gives

$$
{}^R\underset{\sim}{\alpha}_{AB}\cdot\underset{\sim}{u} = -a(2+S_\phi)\alpha_x + 3a\alpha_z
$$

$$
\begin{aligned}
\frac{{}^R d\,\underset{\sim}{u}}{dt} &= \frac{{}^R d}{dt}\Big(\big(\underset{\sim}{r}_{A/E}\times\underset{\sim}{j}\big)\times\underset{\sim}{j}\Big) = \frac{{}^R d}{dt}\Big(\underset{\sim}{j}\times\big(\underset{\sim}{j}\times\underset{\sim}{r}_{A/E}\big)\Big) = \underset{\sim}{j}\times\left(\underset{\sim}{j}\times\frac{{}^R d\,\underset{\sim}{r}_{A/E}}{dt}\right) = \underset{\sim}{j}\times\left(\underset{\sim}{j}\times{}^R\underset{\sim}{v}_A\right) \\
&= \underset{\sim}{j}\times\Big(\underset{\sim}{j}\times a\Omega\big(C_\phi\,\underset{\sim}{i} + S_\phi\,\underset{\sim}{j}\big)\Big) = \underset{\sim}{j}\times\big(-a\Omega C_\phi\,\underset{\sim}{k}\big) = -a\Omega C_\phi\,\underset{\sim}{i}
\end{aligned}
$$

Substituting into the ***differentiated constraint equation*** gives the final equation needed for the acceleration analysis.

$$
-a(2+S_\phi)\alpha_x + 3a\alpha_z = a\Omega C_\phi\omega_x
$$

Combining this result with the ***first*** and ***third*** equations from above gives three equations for the ***angular acceleration*** components. Note again that the parameter "a" has been ***cancelled*** from each equation.

$$\begin{bmatrix} 0 & 3 & (1-C_\phi)-\hat{y}_B \\ \hat{y}_B-(1-C_\phi) & (2+S_\phi) & 0 \\ -(2+S_\phi) & 0 & 3 \end{bmatrix} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = \begin{Bmatrix} -\dot{\Omega}C_\phi + \Omega^2 S_\phi - \omega_y\left(\left(\hat{y}_B-(1-C_\phi)\right)\omega_x + (2+S_\phi)\omega_y\right) - \omega_z\left((2+S_\phi)\omega_z + 3\omega_x\right) \\ \omega_x\left((2+S_\phi)\omega_z + 3\omega_x\right) + \omega_y\left(3\omega_y - \left(\hat{y}_B-(1-C_\phi)\right)\omega_z\right) \\ \Omega C_\phi \omega_x \end{Bmatrix}$$

These equations can be solved to provide the ***angular acceleration components*** of bar AB at any ***arbitrary*** disk angle ϕ. These components can then be used in the ***second*** of the kinematics equations to find the ***acceleration*** of the collar. For example, if $\phi = 0$, then $\hat{y}_B = 6$ and the three equations reduce to

$$\begin{bmatrix} 0 & 3 & -6 \\ 6 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = \begin{Bmatrix} -\dot{\Omega} - \omega_y(6\omega_x + 2\omega_y) - \omega_z(2\omega_z + 3\omega_x) \\ \omega_x(2\omega_z + 3\omega_x) + \omega_y(3\omega_y - 6\omega_z) \\ \Omega\omega_x \end{Bmatrix}$$

Solving gives

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = \frac{-1}{78} \begin{bmatrix} 6 & -9 & 12 \\ -18 & -12 & -36 \\ 4 & -6 & -18 \end{bmatrix} \begin{Bmatrix} -\dot{\Omega} - \omega_y(6\omega_x + 2\omega_y) - \omega_z(2\omega_z + 3\omega_x) \\ \omega_x(2\omega_z + 3\omega_x) + \omega_y(3\omega_y - 6\omega_z) \\ \Omega\omega_x \end{Bmatrix}$$

$$a_B = -3a\alpha_x - 2a\alpha_z + a\Omega^2 + \omega_z(3a\omega_y - 6a\omega_z) - \omega_x(6a\omega_x + 2a\omega_y)$$

Notes:

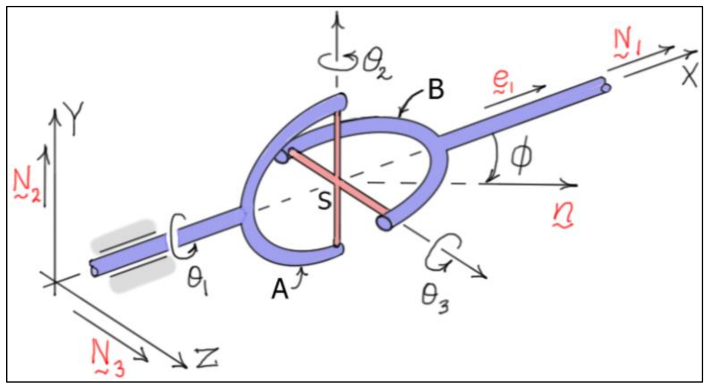
- The equations presented above for an ***arbitrary disk angle*** ϕ are used in ***Unit 10*** to compute ***analytical results*** at a sequence of positions in time. Those results are then ***compared*** to results generated using a MATLAB SimMechanics model.
- As the bar is ***constrained in this example***, there is a ***unique solution*** for its angular velocity components. If, however, the ***revolute*** joint at B is ***replaced*** with a ball and socket joint, the bar would be ***under-constrained***. In that case, there would be ***more unknowns than equations*** indicating that ***multiple solutions*** (motions) are possible. Conversely, if the ball and socket joint at A is ***replaced*** with a joint that

allows only two rotational degrees of freedom (instead of the three allowed by the ball and socket joint), then a ***second constraint equation*** must be written. In that case, there are ***more equations than unknowns*** indicating that there ***may be no solutions*** (meaning motion is not possible).

- The ***point*** of this example is to show that the ***solution process*** used for two-dimensional mechanisms must be ***modified*** for three-dimensional mechanisms to include ***additional constraint equations*** associated with the nature of the interconnecting joints. In the application of this process, it is also clear that ***care must be taken*** in designing the interconnections in three-dimensional mechanisms to ensure that systems are not ***over*** or ***under constrained***. This will ensure that ***motion*** is ***possible*** and that no ***unwanted*** motions can occur.

Example 2: Universal Joint

The diagram depicts a yoke-and-spider ***universal joint***. The joint has three members – the ***input shaft*** *A*, the ***output shaft*** *B*, and the ***spider*** *S* connected with simple revolute joints. The unit vectors $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ represent directions in the ***fixed-frame*** *R*, and the unit vectors $B:\left(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\right)$ represent directions fixed in the ***output shaft*** *B*.



In the figure reference frame *B* is aligned with reference frame *R*, so $\underset{\sim}{e}_i = \underset{\sim}{N}_i \; (i = 1, 2, 3)$. In its final configuration, shaft *B* is aligned with and rotates about the direction indicated by the unit vector $\underset{\sim}{n}$ so that $\boxed{\underset{\sim}{e}_1 = \underset{\sim}{n} = C_\phi \underset{\sim}{N}_1 + S_\phi \underset{\sim}{N}_3}$.

Find:

Using a 1-2-3 ***orientation angle sequence*** to relate the orientation of output shaft *B* to the base frame *R*, find an ***expression*** that relates the speeds of the input and output shafts at any ***arbitrary*** input angle θ_1.

Solution:

Geometric Preliminaries:

Given the setup shown in the diagram, the ***orientation*** of the output shaft *B* can be described using a 1-2-3 orientation angle sequence. The first rotation (θ_1) describes the rotation of the input shaft *A* about its axis, the second rotation (θ_2) describes the rotation of the spider *S* relative to shaft *A*, and the final rotation (θ_3) describes the rotation of shaft *B* relative to spider *S*. Each of these is a simple rotation about the directions indicated in the diagram. Using the results presented in Unit 5 for a 1-2-3 orientation angle sequence, the unit vectors in shaft B can be written in terms of the unit vectors in the ground frame as follows

$$\begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = \begin{bmatrix} C_2C_3 & C_1S_3+S_1S_2C_3 & S_1S_3-C_1S_2C_3 \\ -C_2S_3 & C_1C_3-S_1S_2S_3 & S_1C_3+C_1S_2S_3 \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix}$$

Using the known orientation of $\underset{\sim}{e}_1$ along with the first of these equations gives ***three scalar equations*** that relate the orientation angles to the shaft offset angle ϕ.

$$\begin{aligned} \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_1 &= C_\phi = C_2C_3 \\ \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_2 &= 0 \quad = C_1S_3+S_1S_2C_3 \\ \underset{\sim}{e}_1 \cdot \underset{\sim}{N}_3 &= S_\phi = S_1S_3-C_1S_2C_3 \end{aligned}$$

The ***first*** of these equations provides a simple expression for C_ϕ, and the ***second*** and ***third*** equations can be solved to provide a simple expression for S_ϕ. Specifically, multiplying the second equation by C_1 and the third equation by S_1 and adding the equations gives

$$S_1S_\phi = C_1^2S_3 + \cancel{C_1S_1S_2C_3} + S_1^2S_3 - \cancel{S_1C_1S_2C_3} = \left(C_1^2+S_1^2\right)S_3 = S_3$$

Final geometric results: $\boxed{C_\phi = C_2C_3}$ and $\boxed{S_\phi = S_3/S_1}$

Kinematic Analysis:

Again, using the results from Unit 5 for a 1-2-3 orientation angle sequence, the ***angular velocity*** of the output shaft B can be written as

$${}^R\underset{\sim}{\omega}_B = \omega_1\,\underset{\sim}{e}_1 + \omega_2\,\underset{\sim}{e}_2 + \omega_3\,\underset{\sim}{e}_3 = \left(\dot{\theta}_1C_2C_3+\dot{\theta}_2S_3\right)\underset{\sim}{e}_1 + \left(-\dot{\theta}_1C_2S_3+\dot{\theta}_2C_3\right)\underset{\sim}{e}_2 + \left(\dot{\theta}_1S_2+\dot{\theta}_3\right)\underset{\sim}{e}_3$$

Since shaft B rotates only about the $\underset{\sim}{e}_1$ direction, the following ***three scalar equations*** are obtained

$$\begin{aligned} \omega_1 &= \dot{\theta}_1C_2C_3+\dot{\theta}_2S_3 = \omega_B \\ \omega_2 &= -\dot{\theta}_1C_2S_3+\dot{\theta}_2C_3 = 0 \\ \omega_3 &= \dot{\theta}_1S_2+\dot{\theta}_3 = 0 \end{aligned}$$

Multiplying the first of these equations by C_3 and the second by S_3 and subtracting the equations gives

$$C_3\omega_B = \dot{\theta}_1C_2C_3^2 + \cancel{\dot{\theta}_2S_3C_3} + \dot{\theta}_1C_2S_3^2 - \cancel{\dot{\theta}_2C_3S_3} = \dot{\theta}_1C_2\underbrace{\left(C_3^2+S_3^2\right)}_{=1} = \dot{\theta}_1C_2$$

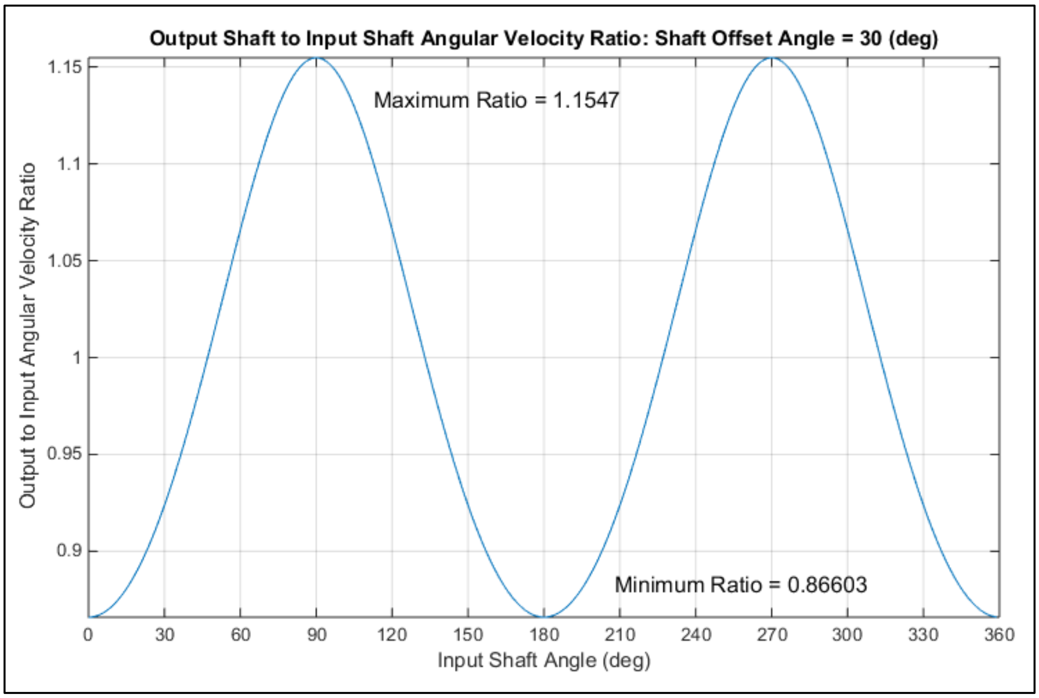
$$\Rightarrow \quad \boxed{\omega_B = \left(\frac{C_2}{C_3}\right)\dot{\theta}_1}$$

This result can be converted into the final result by using the results from the geometric analysis.

$$\boxed{\omega_B = \left(\frac{C_2}{C_3}\right)\dot{\theta}_1 = \left(\frac{C_2 C_3}{C_3^2}\right)\dot{\theta}_1 = \left(\frac{C_\phi}{1-S_3^2}\right)\dot{\theta}_1 = \left(\frac{C_\phi}{1-S_1^2 S_\phi^2}\right)\dot{\theta}_1} \quad \Rightarrow \quad \boxed{\omega_B = \left(\frac{\cos(\phi)}{1-\sin^2(\theta_1)\sin^2(\phi)}\right)\dot{\theta}_1}$$

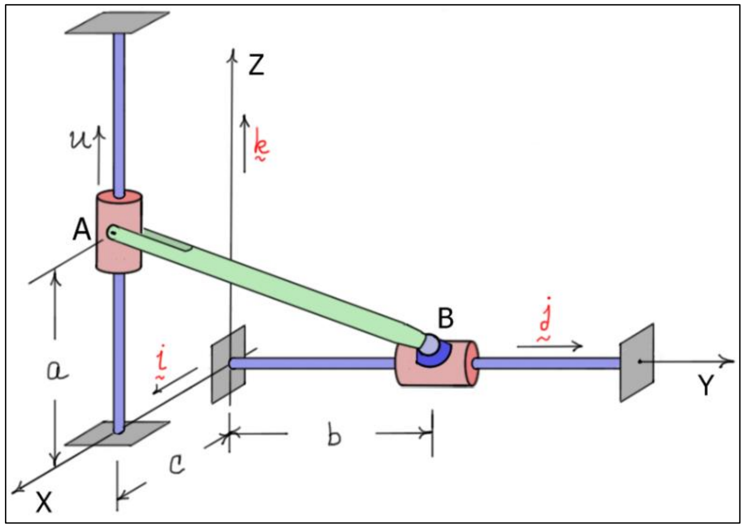
The final boxed equations give the output shaft angular velocity as a function of the shaft offset angle ϕ and the input shaft's angular position (θ_1) and angular velocity $(\dot{\theta}_1)$. Note if $\phi = 0$, then the two shafts have the ***same angular speed***, and if $\phi = 90\ (\text{deg})$, then the output shaft ***cannot rotate***.

The angular velocity ***ratio*** $\omega_B / \dot{\theta}_1$ is ***plotted*** in the graph below for a shaft offset angle of $\phi = 30\ (\text{deg})$. Note that as the input shaft angle rotates through 360 degrees, the output shaft goes through ***two cycles*** of ***speeding up*** and ***slowing down*** relative to the input shaft. So, even when the input shaft is rotating at a ***constant angular rate***, the output shaft is ***not***.



Exercises:

8.1 The system shown consists of ***bar*** *AB* whose ends are connected to ***collars*** that slide along the two fixed poles. The collar at *B* can ***only translate*** along the horizontal bar (***prismatic joint***), while the collar at *A* can both ***translate*** and ***rotate*** relative to the vertical bar (***cylindrical joint***). The bar is connected to the collar at *B* using a ***ball and socket*** joint, and it is connected to the collar at *A* using a ***pin joint***. Find: a) ${}^R\underset{\sim}{v}_B$ the velocity of *B* and ${}^R\underset{\sim}{\omega}_{AB}$ the angular velocity of *AB* relative to the ground frame, and b) ${}^R\underset{\sim}{a}_B$ the acceleration of *B* and ${}^R\underset{\sim}{\alpha}_{AB}$ the angular acceleration of *AB* relative to the ground frame. ***Neglect*** the ***size*** of the collars.



Note that the pin (revolute) joint at *A* allows rotation of the bar relative to the collar in the direction of $\underset{\sim}{r}_{B/A} \times \underset{\sim}{k}$.

Answers:

$${}^R\underset{\sim}{v}_B = -\left(\tfrac{a}{b}\right) u\, \underset{\sim}{j} \qquad \omega_x = \frac{-b\,u}{b^2+c^2} \qquad \omega_y = \frac{-c\,u}{b^2+c^2} \qquad \omega_z = \frac{a\,c\,u}{b\left(b^2+c^2\right)}$$

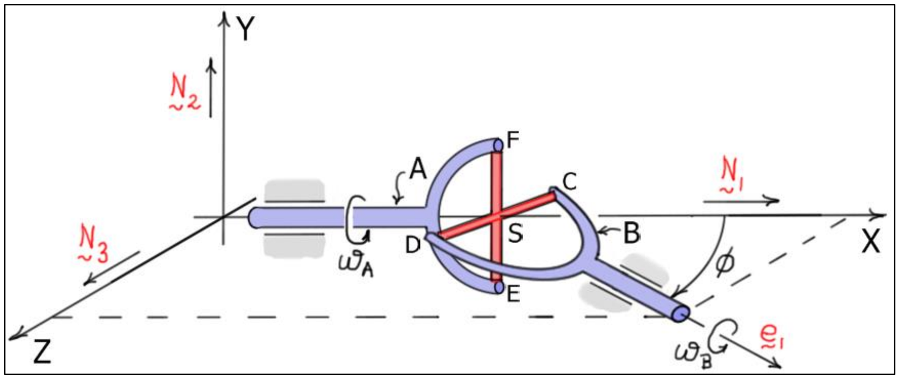
$$\begin{Bmatrix} \alpha_x \\ \alpha_y \end{Bmatrix} = \frac{1}{b^2+c^2} \begin{bmatrix} -b & c \\ -c & -b \end{bmatrix} \begin{Bmatrix} \dot{u} - \omega_x\left(c\,\omega_z - a\,\omega_x\right) + \omega_y\left(a\,\omega_y + b\,\omega_z\right) \\ -\left(a\omega_y + b\omega_z\right)\omega_x - \left(-a\omega_x + c\omega_z\right)\omega_y \end{Bmatrix}$$

$$\alpha_z = \tfrac{1}{b}\left(\omega_y\left(b\,\omega_x + c\,\omega_y\right) + \omega_z\left(c\,\omega_z - a\,\omega_x\right) - a\alpha_y\right)$$

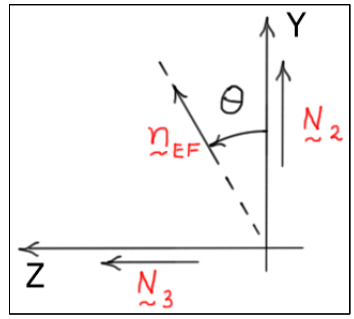
$$a_B = a\,\alpha_x - c\,\alpha_z - \omega_z\left(a\,\omega_y + b\,\omega_z\right) - \omega_x\left(b\,\omega_x + c\,\omega_y\right)$$

8.2 The diagram below depicts a yoke-and-spider ***universal joint***. The joint has three members – the ***input shaft*** *A*, the ***output shaft*** *B*, and the ***spider*** *S*. Shaft *A* rotates about the *X*-axis, and shaft *B* rotates about the $\underset{\sim}{e}_1$ direction which is in the *XZ* plane and makes an angle of ϕ with the *X*-axis. ***At the instant shown***, spider *S* rotates ***relative*** to shaft *A* about the direction of spider segment *EF* (represented by unit vector $\underset{\sim}{n}_{EF} = \underset{\sim}{N}_2$), and it rotates ***relative*** to shaft *B* about the spider segment *CD* (represented by unit vector $\underset{\sim}{n}_{CD} = \underset{\sim}{e}_1 \times \underset{\sim}{n}_{EF} = \underset{\sim}{e}_1 \times \underset{\sim}{N}_2$). ***At the instant shown***, find ω_B / ω_A the ratio of the speed of shaft *B* to the speed of shaft *A*. Hint: Apply the ***summation rule*** for angular velocities (as presented in Unit 1) through the joint using the known directions for the angular velocities and relative angular velocities.

Answer: $\boxed{\omega_B/\omega_A = \cos(\phi)}$



8.3 Referring again to the diagram of the yoke-and-spider ***universal joint*** of Exercise 8.2, let input shaft *A* rotate through an angle θ so the spider segment *EF* is at an angle θ to the *Y*-axis, so $\underset{\sim}{n}_{EF} = C_\theta \underset{\sim}{N}_2 + S_\theta \underset{\sim}{N}_3$. As before, shaft *B* rotates about the $\underset{\sim}{e}_1$ direction, and $\underset{\sim}{n}_{CD} = \underset{\sim}{e}_1 \times \underset{\sim}{n}_{EF}$. Find ω_B/ω_A the ratio of the speed of shaft *B* to the speed of shaft *A* as a function of the input shaft angle θ. Hint: Apply the ***summation rule*** for angular velocities (as presented in Unit 1) ***through the joint*** using the known directions for the angular velocities and relative angular velocities.



Answer: $\boxed{\dfrac{\omega_B}{\omega_A} = \dfrac{\cos(\phi)}{1-\sin^2(\theta)\sin^2(\phi)}}$

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