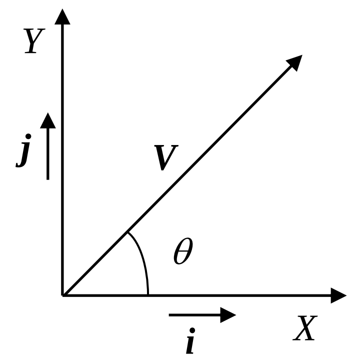
ENGR 1990 Engineering Mathematics
Application of Two-Dimensional Vectors – ME 2560, ME 2570, ME 2580

Scalars and Vectors

A ***scalar*** is a quantity represented by a ***positive or negative number***. It contains a magnitude (its absolute value) and a sign. They are sometimes called ***one-dimensional vectors***, because the sign refers to the direction along a single axis. Examples include length, area, volume, mass, pressure, and temperature.

A two-dimensional vector is represented by a ***magnitude*** and a ***direction*** related to ***two reference axes***. Usually, the reference axes (X and Y) are perpendicular to each other. In application, vectors can be categorized as ***fixed*** or ***free***. A fixed vector is defined to be anchored at a specific point, whereas free vectors can be located anywhere without changing their meaning. Whether a vector is thought to be fixed or free depends on the quantity the vector represents.



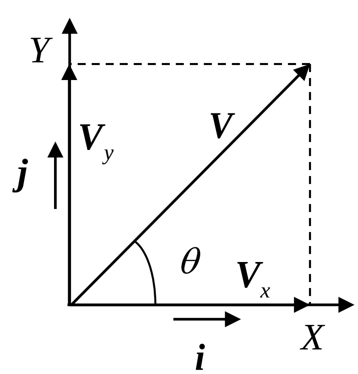
For example, if vector $\boldsymbol{V}$ in the diagram represents a ***force*** acting on some object, it is considered to be a fixed vector, because its point of application is important. In contrast, consider vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ shown in the diagram. These vectors have unit magnitude and point along the X and Y axes, respectively. They are called ***unit vectors*** and are used to define directions of interest. Since their point of origin is not important, they are free vectors. The ***mathematical representation*** of a vector does not indicate whether it is fixed or free, so we must be mindful of this as we use them.

<u>Given:</u> The magnitude $\left(\left|\boldsymbol{V}\right|\right)$ and direction (θ) of vector $\boldsymbol{V}$.

<u>Find:</u> Its X and Y ***components***.

<u>Solution:</u>

The diagram shows the X and Y components of $\boldsymbol{V}$ labeled as $\boldsymbol{V}_x$ and $\boldsymbol{V}_y$. The components are also 2D vectors. Their magnitudes are given by right-triangle trigonometry and their directions are along $\boldsymbol{i}$ and $\boldsymbol{j}$, respectively. Vector $\boldsymbol{V}$ is the sum of these two components.



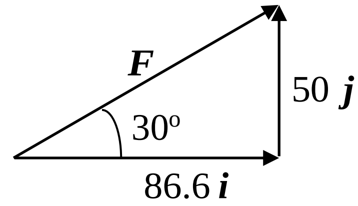
$$\boxed{\boldsymbol{V}_x = \left|\boldsymbol{V}\right|\cos(\theta)\,\boldsymbol{i}} \qquad \boxed{\boldsymbol{V}_y = \left|\boldsymbol{V}\right|\sin(\theta)\,\boldsymbol{j}} \qquad \boxed{\boldsymbol{V} = \left|\boldsymbol{V}\right|\cos(\theta)\,\boldsymbol{i} + \left|\boldsymbol{V}\right|\sin(\theta)\,\boldsymbol{j}}$$

Example 1:

Given: A force $\boldsymbol{F}$ has a magnitude $|\boldsymbol{F}| = 100$ (lbs) and an angle $\theta = 30$ (deg).

Find: Express the force $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

Solution:



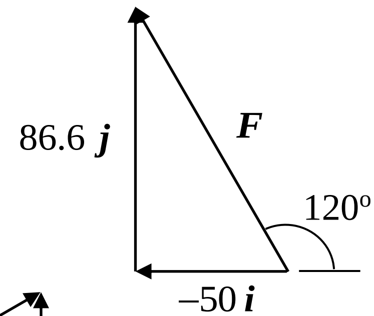
$$\boxed{\boldsymbol{F} = 100\cos(30)\,\boldsymbol{i} + 100\sin(30)\,\boldsymbol{j} = 86.6\,\boldsymbol{i} + 50\,\boldsymbol{j} \text{ (lbs)}}$$

Example 2:

Given: A force $\boldsymbol{F}$ has a magnitude $|\boldsymbol{F}| = 100$ (lbs) and an angle $\theta = 120$ (deg).

Find: Express the force $\boldsymbol{F}$ in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

Solution:

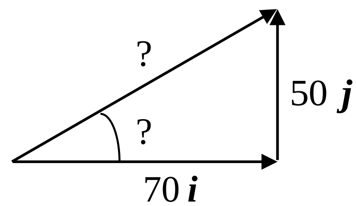


$$\boxed{\boldsymbol{F} = 100\cos(120)\,\boldsymbol{i} + 100\sin(120)\,\boldsymbol{j} = -50\,\boldsymbol{i} + 86.6\,\boldsymbol{j} \text{ (lbs)}}$$

Example 3:

Given: A force $\boldsymbol{F} = 70\,\boldsymbol{i} + 50\,\boldsymbol{j}$ (lbs).

Find: The magnitude and direction of $\boldsymbol{F}$.



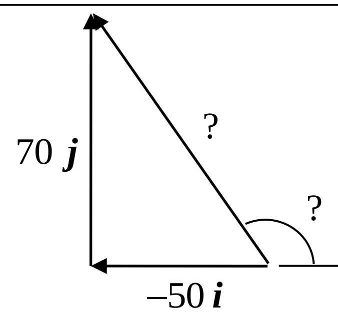
Solution:

$$\boxed{|\boldsymbol{F}| = \sqrt{70^2 + 50^2} = 86.0 \text{ (lbs)}} \quad \text{and} \quad \boxed{\theta = \tan^{-1}(50/70) = \begin{cases} 35.54 \text{ (deg)} \\ 0.6202 \text{ (rad)} \end{cases}}$$

Example 4:

Given: A force $\boldsymbol{F} = -50\,\boldsymbol{i} + 70\,\boldsymbol{j}$ (lbs).

Find: The magnitude and direction of $\boldsymbol{F}$.

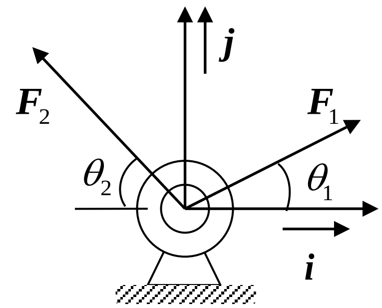


Solution:

$$\boxed{|\boldsymbol{F}| = \sqrt{(-50)^2 + 70^2} = 86.0 \text{ (lbs)}} \quad \boxed{\theta = \tan^{-1}(70/-50) = \begin{cases} -54.46 + 180 = 125.5 \text{ (deg)} \\ -0.9505 + \pi = 2.191 \text{ (rad)} \end{cases}}$$

Vector Addition

To ***add*** two or more vectors, simply express them in terms of the same unit vectors, and then add ***like components***.



Example 5:

Given: $|\boldsymbol{F}_1| = 150$ (lbs), $\theta_1 = 20$ (deg), $|\boldsymbol{F}_2| = 100$ (lbs), $\theta_2 = 60$ (deg).

Find: a) The ***total force*** $\boldsymbol{F}$ acting on the support in terms of the unit vectors shown, and b) the ***magnitude*** and ***direction*** of $\boldsymbol{F}$.

Solution:

a) The total force is the vector sum of the two forces.

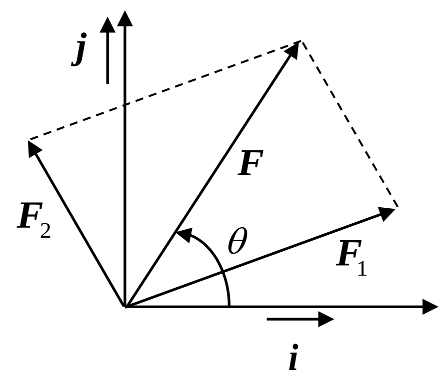
$$\boxed{\boldsymbol{F}_1 = 150\cos(20)\,\boldsymbol{i} + 150\sin(20)\,\boldsymbol{j} = 140.95\,\boldsymbol{i} + 51.3\,\boldsymbol{j} \text{ (lbs)}}$$

$$\boxed{\boldsymbol{F}_2 = -100\cos(60)\,\boldsymbol{i} + 100\sin(60)\,\boldsymbol{j} = -50\,\boldsymbol{i} + 86.6\,\boldsymbol{j} \text{ (lbs)}}$$

$$\boxed{\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = (140.95 - 50)\,\boldsymbol{i} + (51.3 + 86.6)\,\boldsymbol{j} = 90.95\,\boldsymbol{i} + 137.9\,\boldsymbol{j} \text{ (lbs)}}$$

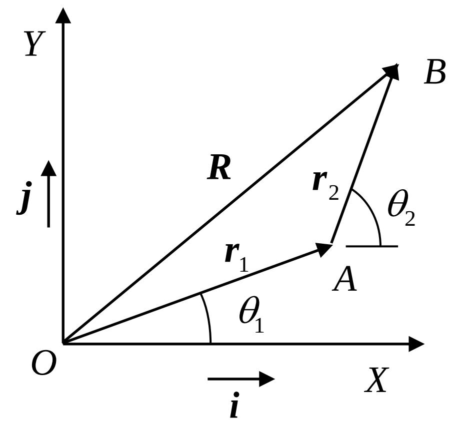
b) $$\boxed{|\boldsymbol{F}| = \sqrt{90.95^2 + 137.9^2} = 165.2 \text{ (lbs)}}$$

$$\boxed{\theta = \tan^{-1}\left(137.9/90.95\right) = \begin{cases} 56.59 \text{ (deg)} \\ 0.9877 \text{ (rad)} \end{cases}}$$



As you can see from the figure, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ form the sides of a parallelogram, and the sum $\boldsymbol{F}$ forms the diagonal. The observation that vectors can be added geometrically in this way is called the ***parallelogram law of addition***. In general, the triangle formed by $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}$ is a non-right triangle. The lengths and angles within this triangle can be studied using the ***laws of cosines*** and ***sines*** as discussed in previous notes.

Example 6:



Given: The lengths and angles of a two link planar robot are $|\boldsymbol{r}_1| = 3$ (ft), $|\boldsymbol{r}_2| = 2$ (ft), $\theta_1 = 20$ (deg), and $\theta_2 = 70$ (deg).

Find: a) The position vector $\boldsymbol{R}$ that defines the position of the end-point of the robot relative to *O*, and b) the magnitude and direction of $\boldsymbol{R}$.

Solution:

a) The position vector $\boldsymbol{R}$ may be calculated by adding the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

$$\begin{aligned} \boldsymbol{R} = \boldsymbol{r}_1 + \boldsymbol{r}_2 &= \left(3\cos(20)\,\boldsymbol{i} + 3\sin(20)\,\boldsymbol{j}\right) + \left(2\cos(70)\,\boldsymbol{i} + 2\sin(70)\,\boldsymbol{j}\right) \\ &= \left(2.8191\,\boldsymbol{i} + 1.0261\,\boldsymbol{j}\right) + \left(0.684\,\boldsymbol{i} + 1.8794\,\boldsymbol{j}\right) \\ &= \boxed{3.503\,\boldsymbol{i} + 2.906\,\boldsymbol{j}\ \text{(ft)}} \end{aligned}$$

b) $\boxed{|\boldsymbol{R}| = \sqrt{3.503^2 + 2.906^2} = 4.55\ \text{(ft)}}$ and $\boxed{\theta = \tan^{-1}\left(2.906/3.503\right) = \begin{cases} 39.7\ \text{(deg)} \\ 0.6925\ \text{(rad)} \end{cases}}$

Scalar (Dot) Product

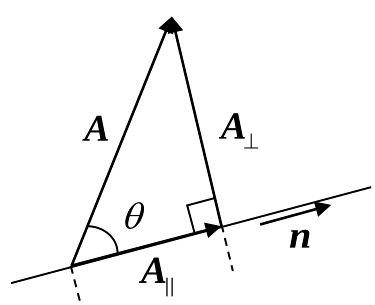
Geometric Definition

The ***scalar*** (or dot) product of two vectors is defines as follows.

$$\boxed{\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos(\boldsymbol{A},\boldsymbol{B})}$$

Here, $\cos(\boldsymbol{A},\boldsymbol{B})$ represents the cosine of the angle between the tails of the two vectors. If one of the vectors is a unit vector, then the scalar product is the ***projection*** of the vector in the direction of the unit vector.

$$\boxed{\boldsymbol{A} \cdot \boldsymbol{n} = |\boldsymbol{A}||\boldsymbol{n}|\cos(\theta) = |\boldsymbol{A}|\cos(\theta)}$$



The ***components*** of $\boldsymbol{A}$ that are ***parallel*** and ***perpendicular*** to $\boldsymbol{n}$ are $\boxed{\boldsymbol{A}_{||} = \left(\boldsymbol{A} \cdot \boldsymbol{n}\right)\boldsymbol{n}}$ and $\boxed{\boldsymbol{A}_\perp = \boldsymbol{A} - \boldsymbol{A}_{||}}$.

Calculation

Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ expressed in terms of a pair of mutually perpendicular unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, we calculate the dot product as

$$\boxed{\boldsymbol{A}\cdot\boldsymbol{B}=\left(a_x\,\boldsymbol{i}+a_y\,\boldsymbol{j}\right)\cdot\left(b_x\,\boldsymbol{i}+b_y\,\boldsymbol{j}\right)=a_xb_x+a_yb_y}$$

The ***dot product*** of two vectors is ***zero*** if they are ***perpendicular*** to each other.

Example 7:

Given: Two vectors, $\boldsymbol{A}=10\boldsymbol{i}+2\boldsymbol{j}$ and $\boldsymbol{B}=3\boldsymbol{i}+7\boldsymbol{j}$

Find: The angle between the two vectors, θ.

Solution:

We can calculate the angle using the inverse cosine function.

$$\boxed{\theta=\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right)=\cos^{-1}\left(\frac{(10\times 3)+(2\times 7)}{\sqrt{10^2+2^2}\sqrt{3^2+7^2}}\right)=\cos^{-1}\left(\frac{44}{77.666}\right)=\begin{cases}55.49\text{ (deg)}\\0.9685\text{ (rad)}\end{cases}}$$

Example 8:

Given: A vector, $\boldsymbol{A}=2\boldsymbol{i}+8\boldsymbol{j}$, and a unit vector $\boldsymbol{n}=\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{j}$.

Find: a) the angle between the two vectors (θ), b) the component of $\boldsymbol{A}$ parallel to $\boldsymbol{n}$, and c) the component of $\boldsymbol{A}$ perpendicular to $\boldsymbol{n}$.

Solution:

a) We can calculate the angle using the inverse cosine function as before.

$$\boxed{\theta=\cos^{-1}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{n}}{|\boldsymbol{A}|}\right)=\cos^{-1}\left(\frac{(2\times\frac{3}{5})+(8\times\frac{4}{5})}{\sqrt{2^2+8^2}}\right)=\cos^{-1}\left(\frac{7.6}{8.2462}\right)=\begin{cases}22.83\text{ (deg)}\\0.3985\text{ (rad)}\end{cases}}$$

b) The component of $\boldsymbol{A}$ parallel to $\boldsymbol{n}$: $\boxed{\boldsymbol{A}_{\|}=(\boldsymbol{A}\cdot\boldsymbol{n})\boldsymbol{n}=7.6\left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{j}\right)=4.56\boldsymbol{i}+6.08\boldsymbol{j}}$

c) The component of $\boldsymbol{A}$ perpendicular to $\boldsymbol{n}$:

$$\boxed{\boldsymbol{A}_{\perp}=\boldsymbol{A}-\boldsymbol{A}_{\|}=\left(2\boldsymbol{i}+8\boldsymbol{j}\right)-\left(4.56\boldsymbol{i}+6.08\boldsymbol{j}\right)=-2.56\boldsymbol{i}+1.92\boldsymbol{j}}$$

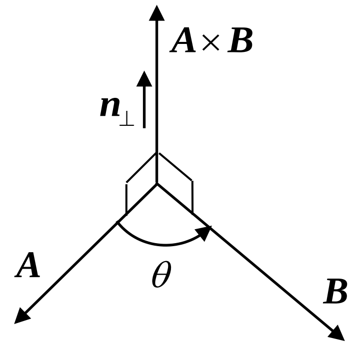
Check: $\boxed{\boldsymbol{A}_{\perp}\cdot\boldsymbol{A}_{\|}=\left(-2.56\boldsymbol{i}+1.92\boldsymbol{j}\right)\cdot\left(4.56\boldsymbol{i}+6.08\boldsymbol{j}\right)=\left(-2.56\times 4.56\right)+\left(1.92\times 6.08\right)=0}$ √

Vector (Cross) Product

<u>Geometric Definition</u>

The ***cross*** product of two vectors is defines as follows.

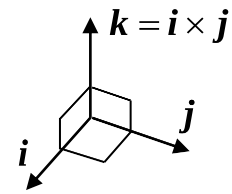
$$\boldsymbol{A}\times\boldsymbol{B}=\left(|\boldsymbol{A}||\boldsymbol{B}|\sin(\boldsymbol{A},\boldsymbol{B})\right)\boldsymbol{n}_\perp$$



Here, $\sin(\boldsymbol{A},\boldsymbol{B})$ is the cosine of the angle between the tails of the two vectors, and $\boldsymbol{n}_\perp$ is a unit vector perpendicular to the plane formed by $\boldsymbol{A}$ and $\boldsymbol{B}$. The sense of $\boldsymbol{n}_\perp$ is defined by the ***right-hand-rule***, that is, the right thumb points in the direction of $\boldsymbol{n}_\perp$ when the fingers of the right hand point from $\boldsymbol{A}$ to $\boldsymbol{B}$.

<u>Calculation</u>

Given two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ expressed in terms of mutually perpendicular unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, we calculate the cross product as



$$\boldsymbol{A}\times\boldsymbol{B}=\left(a_x\,\boldsymbol{i}+a_y\,\boldsymbol{j}\right)\times\left(b_x\,\boldsymbol{i}+b_y\,\boldsymbol{j}\right)=\left(a_xb_y-a_yb_x\right)\boldsymbol{k}$$

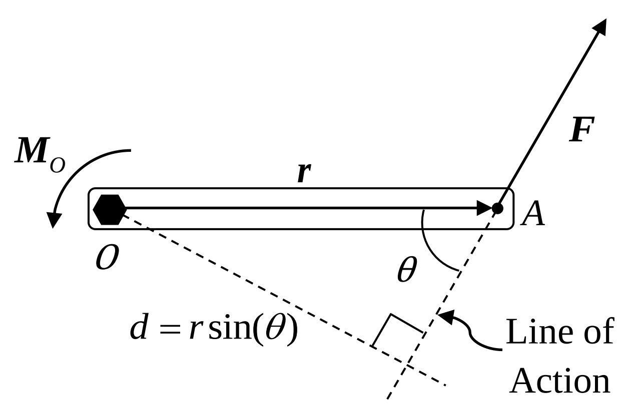
The ***cross product*** of two vectors is ***zero*** if they are ***parallel*** to each other.

This calculation may be calculated from the ***determinant form***. Since the cross product of two dimensional vectors has no $\boldsymbol{i}$ or $\boldsymbol{j}$ components, we write

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_x & a_y & 0\\ b_x & b_y & 0\end{vmatrix}=\left(a_xb_y-a_yb_x\right)\boldsymbol{k}$$

Moment of a Force – Torque

The ***moment*** (or ***torque***) of a force about a point O is defined as the magnitude of the force $\left(|\boldsymbol{F}|\right)$ multiplied by the ***perpendicular distance*** from the point to the ***line of action*** of the force $\left(d=r\sin(\theta)\right)$. The right-hand-rule defines the direction. So, it can be calculated using the ***cross product***.



$$\boxed{\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}}$$

Here, $\boldsymbol{r}$ is a position vector from O to the line of action of $\boldsymbol{F}$.

Example 9:

Given: A force $\boldsymbol{F} = 50\boldsymbol{i} + 100\boldsymbol{j}$ (lbs) is applied at a point A whose coordinates are $(3,2)$ (ft).

Find: a) $\boldsymbol{M}_O$ the moment of the force about the origin O at $(0,0)$, and b) d the perpendicular distance from O to the line of action of the force.

Solution:

a) $$\boxed{\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F} = (3\boldsymbol{i} + 2\boldsymbol{j}) \times (50\boldsymbol{i} + 100\boldsymbol{j}) = ((3\times 100) - (2\times 50))\boldsymbol{k} = 200\boldsymbol{k} \text{ (ft-lbs)}}$$

b) $$\boxed{d = \frac{|\boldsymbol{M}_O|}{|\boldsymbol{F}|} = \frac{200}{\sqrt{50^2 + 100^2}} = \frac{200}{111.80} = 1.79 \text{ (ft)}}$$

Example 10:

Given: A force $\boldsymbol{F} = 50\boldsymbol{i} + 100\boldsymbol{j}$ (lbs) is applied at a point A whose coordinates are $(3,-2)$ (ft).

Find: a) $\boldsymbol{M}_B$ the moment of the force about point B whose coordinates are $(5,10)$ (ft), and b) d the perpendicular distance from B to the line of action of the force.

Solution:

To calculate the moment using the cross product, we must first calculate the position vector that defines the position of A relative to B.

a) $$\boxed{\boldsymbol{r} = \boldsymbol{r}_{A/B} = \boldsymbol{r}_A - \boldsymbol{r}_B = (3\boldsymbol{i} - 2\boldsymbol{j}) - (5\boldsymbol{i} + 10\boldsymbol{j}) = -2\boldsymbol{i} - 12\boldsymbol{j} \text{ (ft)}}$$

$$\boxed{\boldsymbol{M}_B = \boldsymbol{r} \times \boldsymbol{F} = (-2\boldsymbol{i} - 12\boldsymbol{j}) \times (50\boldsymbol{i} + 100\boldsymbol{j}) = ((-2\times 100) + (12\times 50))\boldsymbol{k} = 400\boldsymbol{k} \text{ (ft-lb)}}$$

b) $$\boxed{d = \frac{|\boldsymbol{M}_B|}{|\boldsymbol{F}|} = \frac{400}{\sqrt{50^2 + 100^2}} = \frac{400}{111.80} = 3.58 \text{ (ft)}}$$