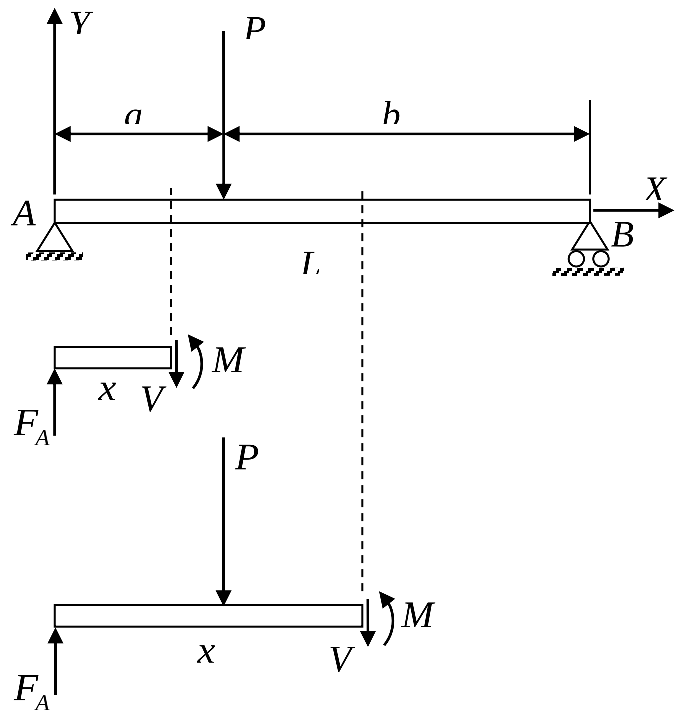
ENGR 1990 Engineering Mathematics
Applications of Derivatives – ME 2560, ME 2570

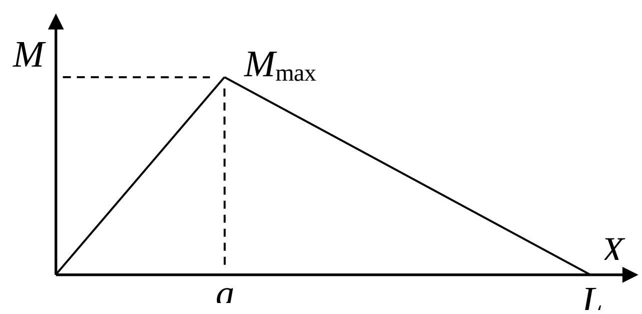
Example #1

Consider a ***long slender beam*** of length L with a ***concentrated load*** P acting at distance a from the left end. Due to this load, the beam experiences an ***internal bending moment*** $M(x)$ and ***internal shearing force*** $V(x)$. As presented in earlier notes, the bending moment is zero at both ends of the beam and rises linearly from there to a maximum value at $x=a$. The shearing force is the derivative of the bending moment.



$$\boxed{V(x)=\frac{dM(x)}{dx}=M'(x)}$$

Given: $P=100$ (lbs), $L=5$ (ft), $a=3.5$ (ft) and $M_{\max}=abP/L$



Find: (a) $M(x)$ for $0\le x\le L$; (b) $V(x)$ for $0\le x\le L$; and (c) plot the functions.

Solution: $M_{\max}=abP/L=(3.5)(1.5)100/5=105$ (ft-lb).

(a) For $(0\le x\le a)$, the slope is $\boxed{m=(105-0)/(3.5-0)=30 \text{ (ft-lb/ft)}}$.

$$\boxed{M(x)=30x \ \text{ (ft-lb)}}$$

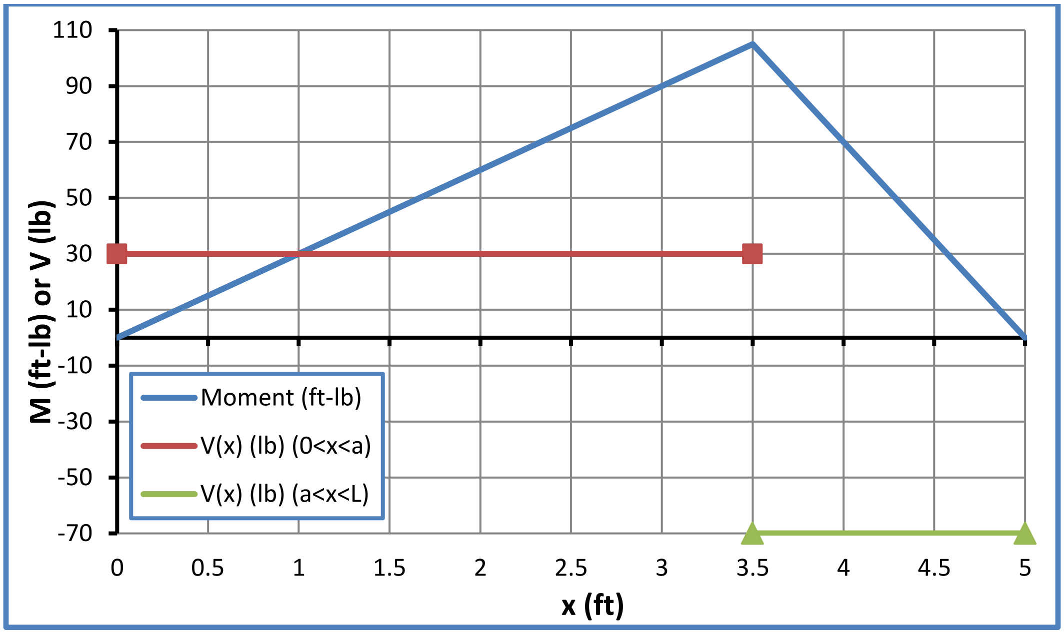
For $(a\le x\le L)$, the slope is $\boxed{m=(0-105)/(5-3.5)=-70 \text{ (ft-lb/ft)}}$. Using the point-slope form

$$(M-105)=-70(x-3.5) \quad\Rightarrow\quad \boxed{M(x)=350-70x \text{ (ft-lb)}}$$

(b) For $\left(0 \le x \le a\right)$, $\boxed{V(x) = M'(x) = \frac{d}{dx}\left(30x\right) = 30 \text{ (lb)}}$

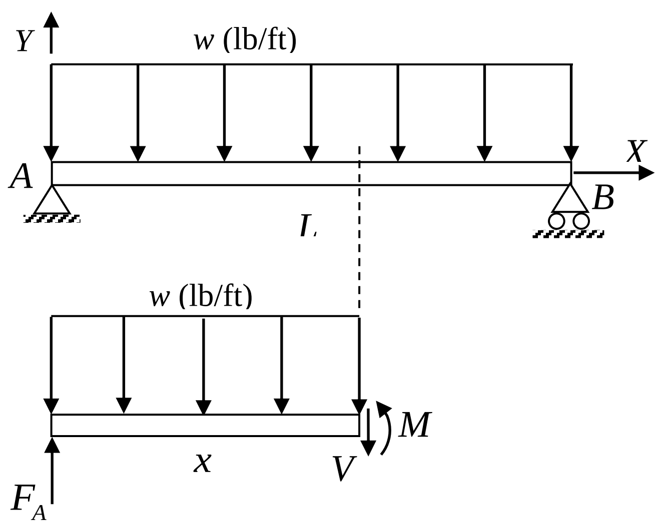
For $\left(a \le x \le L\right)$, $\boxed{V(x) = M'(x) = \frac{d}{dx}\left(350 - 70x\right) = \frac{d}{dx}\left(350\right) + \frac{d}{dx}\left(-70x\right) = -70 \text{ (lb)}}$

(c)



Question: What is the value of $M'(x)$ at $x = 3.5$ (ft)?

Example 2:



Given: $L = 10$ (ft), $w = 100$ (lb/ft), and

$$\boxed{M(x) = 500x - 50x^2 \text{ (ft-lb)}} \quad \left(0 \le x \le L\right)$$

Find: (a) the shearing force $V(x)$; (b) the maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.

Solution:

(a) For $\left(0 \le x \le L\right)$

$$\boxed{V(x) = M'(x) = \frac{d}{dx}\left(500x - 50x^2\right) = \frac{d}{dx}\left(500x\right) + \frac{d}{dx}\left(-50x^2\right) = 500 - 100x \text{ (lb)}}$$

(b) Because the shearing force is continuous, the bending moment is a maximum (or minimum) either at an end of the beam or where the shear ***zero***.

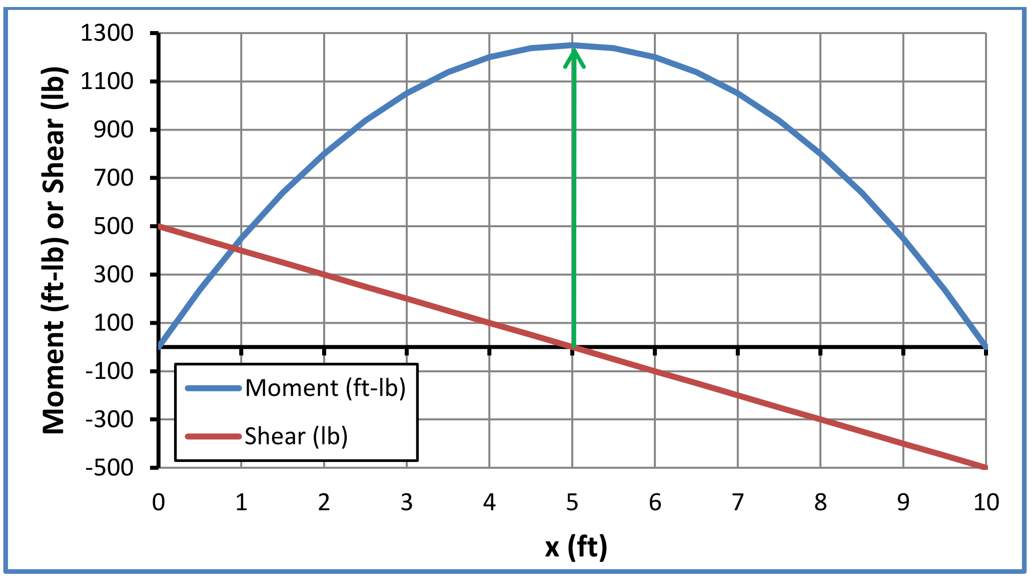
$$V(x) = M'(x) = 500 - 100x = 0 \quad \Rightarrow \boxed{x = 500/100 = 5 \text{ (ft)}}$$

$$\boxed{M(0) = M(L) = 0} \text{ and } \boxed{M(x=5) = (500 \times 5) - (50 \times 5^2) = 1250 \text{ (ft-lb)} = M_{\max}}$$

To verify that it is a maximum, check the sign of $M''(x)$:

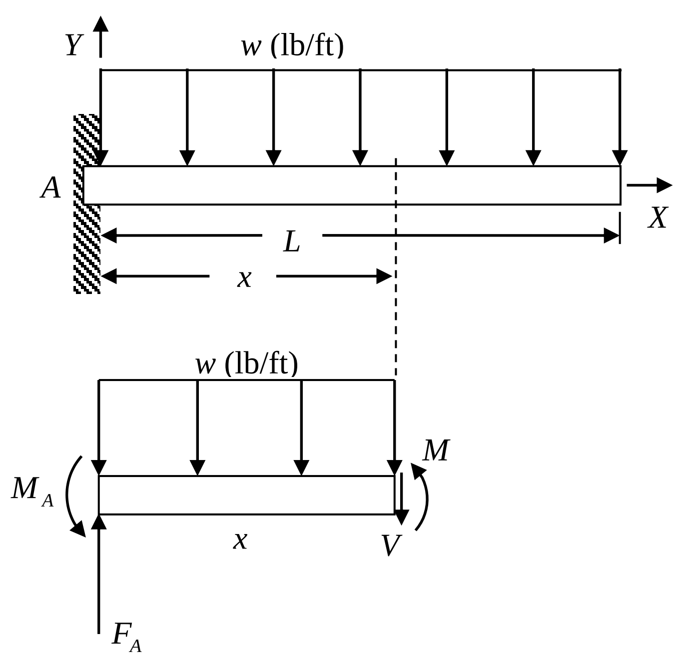
$$\boxed{M''(x) = \frac{d}{dx}(500 - 100x) = -100 < 0} \text{ (it is a maximum)}$$

(c)



<u>Example 3:</u>

Consider a cantilevered beam with a ***uniformly distributed load*** of w (lb/ft). If the beam is cut at a distance x from the wall, we expose the internal ***shearing force*** V and ***bending moment*** M.



<u>Given:</u> $L = 10$ (ft), $w = 100$ (lb/ft), and

$$\boxed{M(x) = -\tfrac{1}{2}wx^2 + wLx - \tfrac{1}{2}wL^2 \text{ (ft-lb)}}$$

Find: (a) the shearing force $V(x)$; (b) the maximum bending moment and its location; and (c) plot $M(x)$ and $V(x)$.

Solution: Using the values for L and w, $\boxed{M(x) = -50x^2 + 1000x - 5000 \text{ (ft-lb)}}$

(a) $\boxed{V(x) = M'(x) = \dfrac{d}{dx}\left(-50x^2 + 1000x - 5000\right) = 1000 - 100x \text{ (lb)}}$

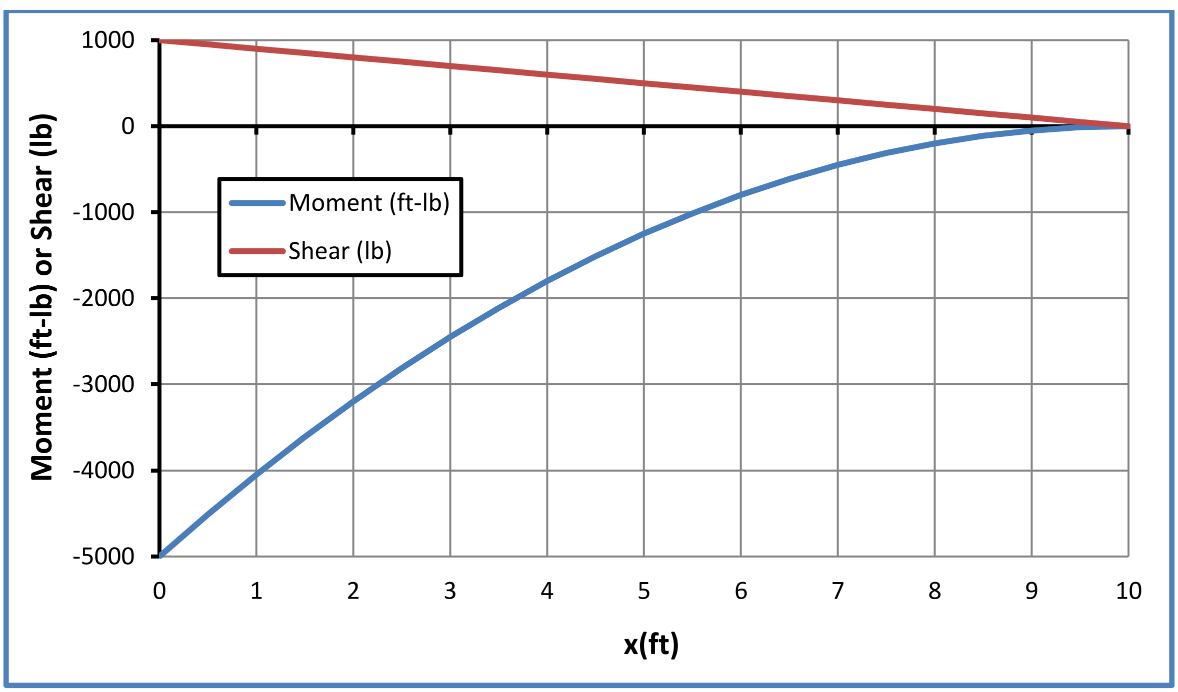
(b) Again, the shearing force is continuous, so the bending moment is a maximum (or minimum) either at an end of the beam or where the shear ***zero***.

$$V(x) = M'(x) = 1000 - 100x = 0 \quad \Rightarrow \boxed{x = 1000/100 = 10 \text{ (ft)}} \text{ (at the end)}$$

$$\boxed{M(0) = -5000 \text{ (ft-lb)}} \quad \boxed{M(10) = 0 \text{ (ft-lb)}} \quad \Rightarrow \boxed{M_{\max} = -5000 \text{ (ft-lb)}}$$

In this case, the maximum occurs at the end of the beam, and not where $M'(x) = 0$, because our concern is with the absolute value of the bending moment. We must design the beam to withstand 5000 (ft-lb) of bending moment, not zero.

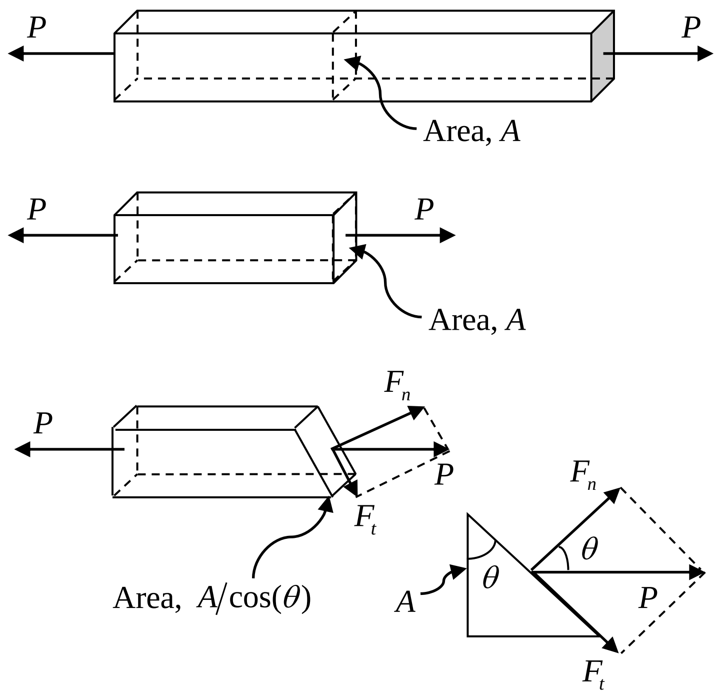
(c)



Example 4:

Consider a bar with rectangular cross-sectional area A and applied force P as shown. Because A is perpendicular (or normal) to the direction of P, the material experiences normal stress only and is defined as

$$\boxed{\sigma = P/A}$$



Now consider a plane at an angle θ to the vertical. Since this plane is not normal to P, the material along this plane experiences both normal stress and shear stress.

The normal stress σ is defined (as before) by the area and the normal force. The shear stress τ is defined by the area and the tangential force.

$$\boxed{\sigma = F_n/\left(A/\cos(\theta)\right) = \left(P\cos(\theta)\right)/\left(A/\cos(\theta)\right) = \left(P/A\right)\cos^2(\theta)}$$

$$\boxed{\tau = F_t/\left(A/\cos(\theta)\right) = \left(P\sin(\theta)\right)/\left(A/\cos(\theta)\right) = \left(P/A\right)\sin(\theta)\cos(\theta)}$$

In a simple tension test, such as that described above, brittle materials tend to fail due to excessive normal stress, and ductile material tend to fail due to excessive shear stress.

By thinking of the normal and shear stresses as functions of the angle θ, we can find which planes experience the ***highest*** normal and shear stresses. We can find maxima and minima by setting $d\sigma/d\theta = 0$ and $d\tau/d\theta = 0$ and solving for the angle θ. Using the product and chain rules gives

$$\boxed{d\sigma/d\theta = \frac{d}{d\theta}\left[\left(P/A\right)\cos^2(\theta)\right] = \left(P/A\right)\left(2\cos(\theta)\right)\left(-\sin(\theta)\right) = -\left(2P/A\right)\sin(\theta)\cos(\theta)}$$

$$\boxed{d^2\sigma/d\theta^2 = \frac{d}{d\theta}\left[-\left(2P/A\right)\sin(\theta)\cos(\theta)\right] = \left(2P/A\right)\left(\sin^2(\theta) - \cos^2(\theta)\right)}$$

$$\boxed{d\tau/d\theta = \frac{d}{d\theta}\left[\left(P/A\right)\sin(\theta)\cos(\theta)\right] = \left(P/A\right)\left[\cos^2(\theta) - \sin^2(\theta)\right] = \left(P/A\right)\cos(2\theta)}$$

$$\boxed{\begin{aligned} d^2\tau/d\theta^2 &= \frac{d}{d\theta}\Big[\big(P/A\big)\cos(2\theta)\Big] = \big(P/A\big)\Big[\big(-\sin(2\theta)\big)\big(2\big)\Big] \\ &= \big(-2P/A\big)\sin(2\theta) \end{aligned}}$$

Setting the derivatives to zero and considering the range of θ is $\boxed{0 \le \theta < \pi/2}$, we get the following results.

Stress	Angle, θ	1^{st} Derivative	2^{nd} Derivative	Type
σ	0	0	negative	maximum
τ	$\pi/4$ (rad) $= 45^{\circ}$	0	negative	maximum

So, ***brittle materials*** will be more likely to fail on a plane ***normal to*** the load, and ***ductile materials*** will be more likely to fail on a ***45º plane***.