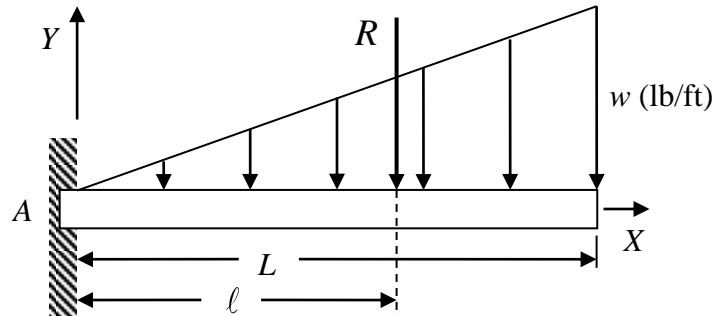


**ENGR 1990 Engineering Mathematics**  
**Applications of Integration in ME 2560, ME 2570**

Example 1: Resultant of a distributed load

The diagram shows a cantilevered beam with a linearly varying load intensity. The maximum load intensity is  $w$  (lb/ft) at the right end of the beam. To calculate the support forces at A, the distributed load can be replaced by a single resultant load  $R$  acting at a distance  $\ell$  from the left end.



The load  $R$  and distance  $\ell$  are found by solving the following equations

$$\boxed{R = \int_0^L w x dx} \quad \text{and} \quad \boxed{R \times \ell = \int_0^L (w x) x dx = \int_0^L w x^2 dx} \quad (1)$$

The first of Eq. (1) equates the resultant *force*  $R$  with the *summation* of the *load intensity*, and the second one equates the *moment* of  $R$  about A with the *sum* of the *moments* of the of the *load intensity*.

Given:  $L = 10$  (ft),  $w = 100$  (lb/ft)

Find: (a) the resultant force  $R$ ; and (b) the distance  $\ell$  that it acts from the support.

Solution:

(a) The resultant load is

$$\boxed{R = \int_0^{10} 10x dx = (5x^2) \Big|_0^{10} = 5 \times 100 = 500 \text{ (lb)}} \quad \text{or} \quad \boxed{R = \int_0^{10} 10x dx = \frac{1}{2} \times 10 \times 100 = 500 \text{ (lb)}}$$

(b) The distance  $\ell$  is

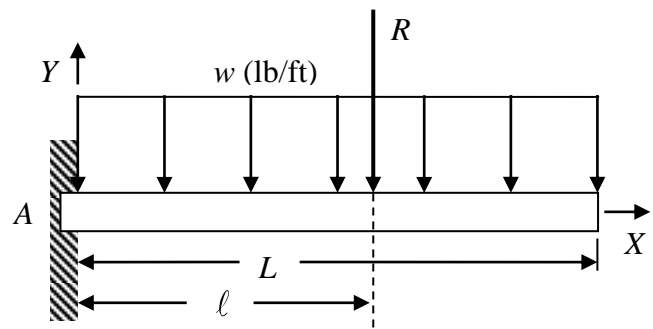
$$\boxed{500 \ell = \int_0^{10} 10x^2 dx = \left(\frac{10}{3} x^3\right) \Big|_0^{10} \Rightarrow \ell = \frac{10 \times 1000}{3 \times 5000} = \frac{2}{3}(10) = 6.\bar{6} \text{ (ft)}}$$

So, for a linearly varying load (starting at zero), the resultant acts  $\frac{2}{3}$  of the way along the distributed load. If the load started at  $w$  (lb/ft) at the wall, then the resultant would be located  $\frac{1}{3}$  of the way along the load.

Example 2:

Given:  $L = 10$  (ft),  $w = 100$  (lb/ft)

Find: (a) the resultant force  $R$ ; and (b) the distance  $\ell$  that it acts from the support.



Solution:

(a) The resultant load is

$$R = \int_0^{10} 100 dx = 10 \times 100 = 1000 \text{ (lb)}$$

(b) The distance  $\ell$  is

$$1000 \ell = \int_0^{10} 100x dx = \frac{1}{2} \times 10 \times 1000 \Rightarrow \ell = \frac{5000}{1000} = \frac{1}{2}(10) = 5 \text{ (ft)}$$

So, for a uniformly distributed load, the resultant acts  $\frac{1}{2}$  of the way along the load.

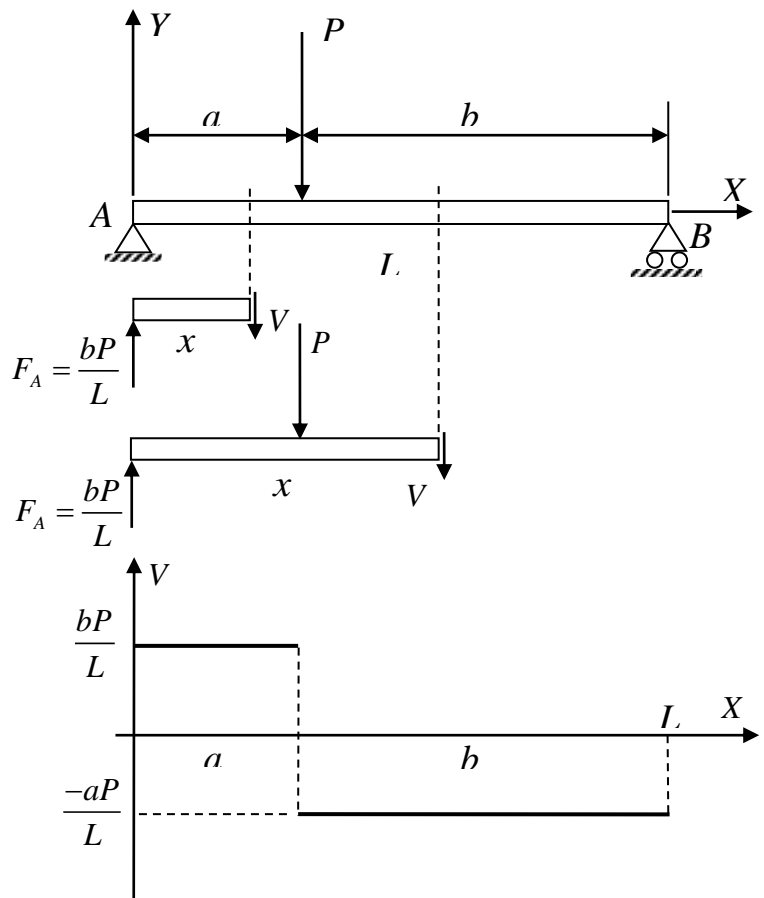
Example 3:

The diagram of the internal shearing force for the simply supported beam with a concentrated load is shown. The internal bending moment is related to the shearing force by the equation

$$M(x) = \int V(x) dx$$

Given:  $P = 100$  (lbs),  $L = 5$  (ft),  
 $a = 3.5$  (ft),  $b = 1.5$  (ft), and  
 $M(0) = M(L) = 0$

Find: (a) the moment diagram for the beam; and (b)  $M_{\max}$  the maximum bending moment in the beam.



Solution:

(a) We can construct the moment diagram from the shear diagram. Where the shear is *constant*, the moment varies *linearly* with  $x$ .

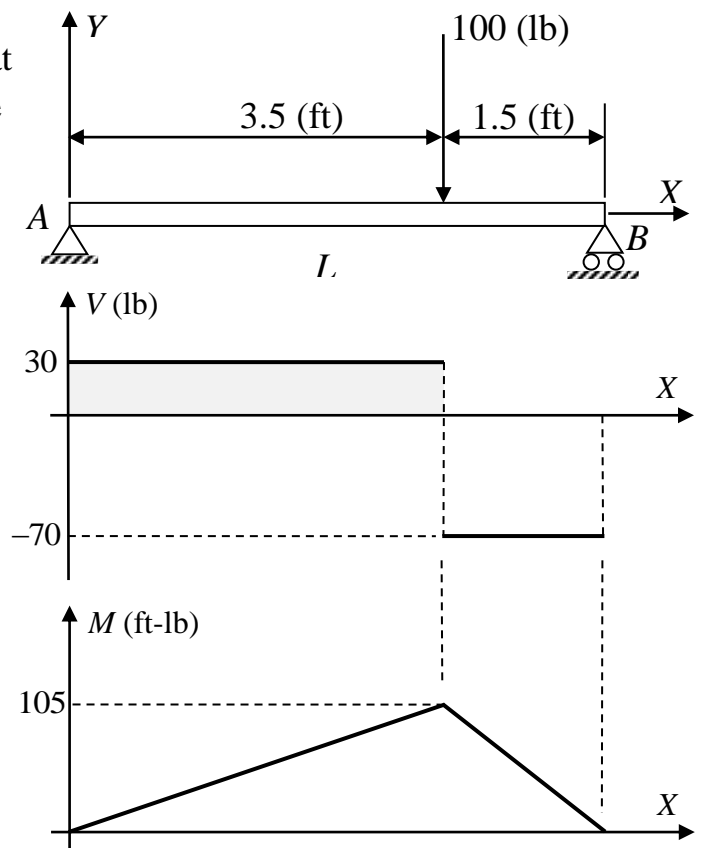
$$\boxed{V_a = bP/L = 1.5 \times 100/5 = 30 \text{ (lb)}} \quad \text{and} \quad \boxed{V_b = -aP/L = -3.5 \times 100/5 = -70 \text{ (lb)}}$$

$$\boxed{M(x) = \int 30 dx = 30x + D = 30x} \quad \text{for } 0 \leq x \leq 3.5 \text{ (recall that } M(0) = 0)$$

$$\boxed{M(x) = \int -70 dx = -70x + D = -70x + 350} \quad \text{for } 3.5 \leq x \leq 5 \text{ (} M(3.5) = 105 \text{ (ft-lb))}$$

(b) The maximum bending moment occurs at the concentrated load. It is the area under the shear diagram from 0 to 3.5 feet.

$$\boxed{M_{\max} = 30 \times 3.5 = 105 \text{ (ft-lb)}}$$



Example 4:

Given:  $L = 10$  (ft),  $w = 100$  (lb/ft),  
 $M(0) = M(L) = 0$ , and

$$V(x) = 500 - 100x \text{ (lb)} \quad (0 \leq x \leq L)$$

Find: (a) the bending moment diagram;  
 and (b) the maximum bending moment.

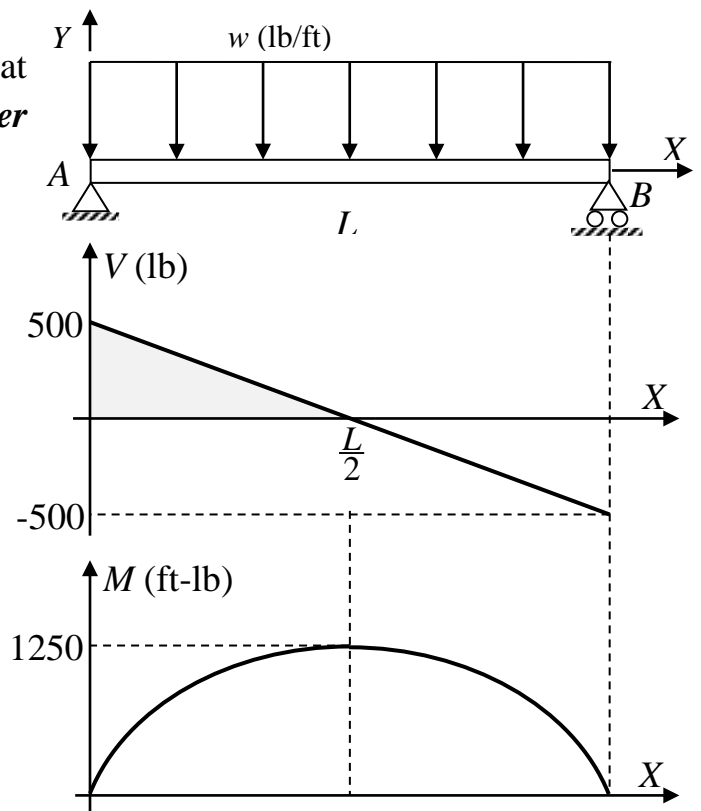
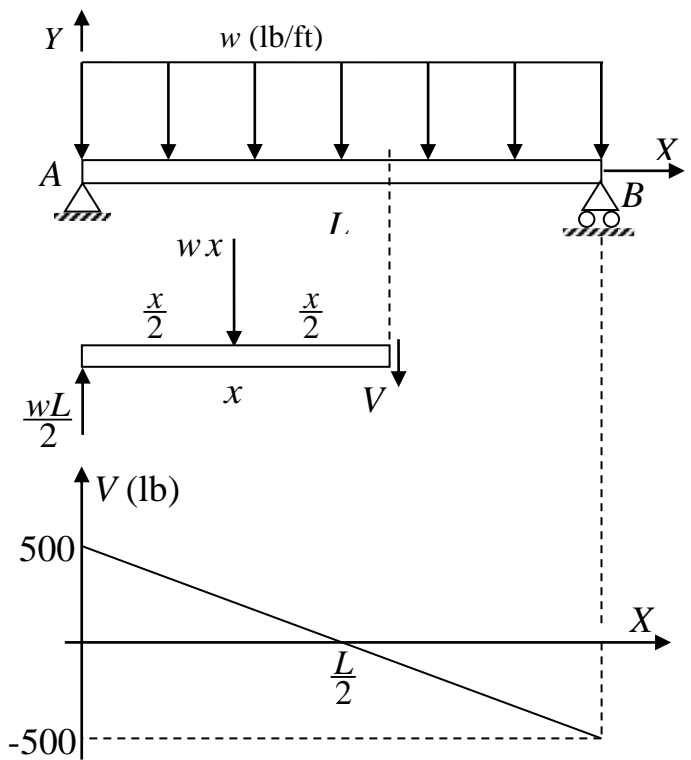
Solution:

(a) Again, we can construct the moment diagram from the shear diagram. Where the shear varies *linearly*, the moment varies *quadratically* with  $x$ .

$$\begin{aligned} M(x) &= \int (500 - 100x) dx \\ &= 500x - 50x^2 + D \quad (M(0) = 0) \\ &= 500x - 50x^2 \text{ (ft-lb)} \end{aligned}$$

(b) The maximum bending moment occurs at the midpoint of the beam. It is the *area under the shear diagram* from 0 to  $L/2$ .

$$M_{\max} = \frac{1}{2} \times 5 \times 500 = 1250 \text{ (ft-lb)}$$



Example 5:

Given:  $L = 10$  (ft),  $w = 100$  (lb/ft),

$M(L) = 0$ , and

$$V(x) = 1000 - 100x \text{ (lb)} \quad (0 \leq x \leq L)$$

Find: (a) the bending moment diagram; and (b) the maximum bending moment.

Solution:

(a) In this case, the shear varies **linearly** with  $x$ , so the moment will vary **quadratically** with  $x$ . Given  $M(10) = 0$ , we find

$$\begin{aligned} M(x) &= \int (1000 - 100x) dx \\ &= 1000x - 50x^2 + D \\ &= -5000 + 1000x - 50x^2 \text{ (ft-lb)} \end{aligned}$$

(b) The maximum moment occurs at the left end of the beam and is equal to the area under the shear diagram from 0 to  $L$ . Why?

$$M_{\max} = \frac{1}{2} \times 10 \times 1000 = 5000 \text{ (ft-lb)}$$

