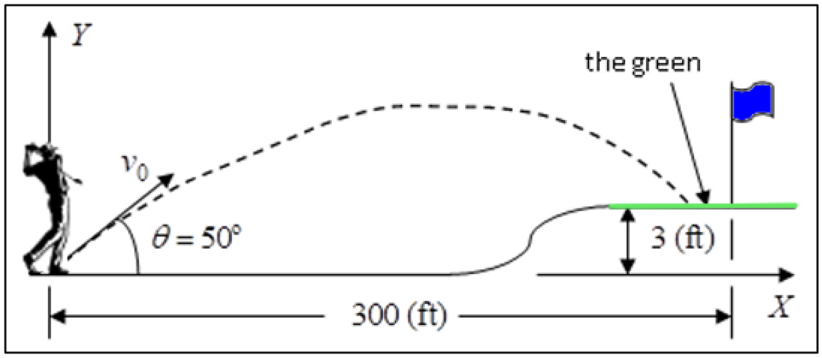
ENGR 1990 Engineering Mathematics
Application of Quadratic Equations – ME 2580 Dynamics

Example #1

Given: A ***golfer*** hits a ball with an ***initial velocity*** of $\mathbf{v}_0 = 96 \text{ (ft/s)}$ at an ***angle*** of $\theta = 50 \text{ (deg)}$. If we neglect air resistance, the following equations describe the x and y positions of the ball (measured in feet) as a function of time (t, sec). That is, they describe the ***path*** of the ball.



$$\boxed{x(t) = 61.71t} \qquad \boxed{y(t) = 73.54t - 16.1t^2} \qquad (1)$$

Find: a) the times when $y = 50 \text{ (ft)}$, b) how long it takes for the ball to land on the green, c) the x-coordinate of the ball when it lands on the green, d) the maximum height, e) the time it takes for the ball to reach $y = 100 \text{ (ft)}$, and f) the quadratic function $y = f(x)$.

Solution:

a) If $y = 50 \text{ (ft)}$, then we can write the following quadratic equation for $y(t)$

$$\boxed{16.1t^2 - 73.54t + 50 = 0} \qquad (2)$$

Besides using our calculator directly, there are ***three*** basic ***analytical methods*** for finding the ***roots*** of any quadratic equation – the ***quadratic formula***, ***completing the square***, and ***factoring***.

Method 1: Using the ***quadratic formula***, $\boxed{t_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$, we find the roots to be

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)50}}{2(16.1)} = 2.2839 \pm 1.4527 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.8312 \approx 0.831 \text{ (s)} \\ 3.7366 \approx 3.74 \text{ (s)} \end{cases}}$$

Method 2: To ***complete the square*** of the quadratic equation, we first divide through by the coefficient of t^2 to get, $\boxed{t^2 - 4.5677t + 3.1056 = 0}$. Then, we complete the square as follows:

$$t^2 - 4.5677t + 3.1056 = \left(t - \frac{4.5677}{2}\right)^2 + 3.1056 - \left(\frac{4.5677}{2}\right)^2 = \left(t - 2.2838\right)^2 - 2.1101$$

or

$$(t - 2.2838)^2 = 2.1101 \quad \Rightarrow \quad t - 2.2838 = \pm 1.4526 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.8312 \approx 0.831 \text{ (s)} \\ 3.7364 \approx 3.74 \text{ (s)} \end{cases}}$$

Method 3: The final method is factoring. ***Factoring*** is useful if the roots of the quadratic are integers. In this case, as in many real engineering problems, they are not. However, knowing the roots from either of the above methods, we recognize that

$$\boxed{\begin{gathered}(t - 0.8312)(t - 3.7364) = t^2 - (0.8312 + 3.7364)t + (0.8312 \times 3.7364) \\ \approx t^2 - 4.5677t + 3.1056\end{gathered}}$$

b) The ball ***hits the green*** when $y = 3$ (ft), so we can write $\boxed{16.1t^2 - 73.54t + 3 = 0}$. Using the quadratic formula,

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)3}}{2(16.1)} = 2.2839 \pm 2.2427 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.0411 \text{ (s)} \\ 4.5265 \approx 4.53 \text{ (s)} \end{cases}}$$

Clearly, the second of these two times is the answer for which we are looking.

c) We can find out how close the ball lands to the pin by calculating the the *x*-coordinate of the ball when it hits.

$$\boxed{x(t)\big|_{t=4.5265} = 61.71(4.5265) = 279.33 \approx 279 \text{ (ft)}} \qquad (3)$$

So, the ball ***lands*** about 21 (ft) ***short of the pin***.

d) The ***maximum height*** of the ball occurs at the midpoint of the ball's flight at $t = 2.2839\text{ (s)}$, so

$$\boxed{y_{\max} = y(t)\big|_{t=2.2839} = \left(73.54\times 2.2839\right) - \left(16.1\times 2.2839^2\right) = 83.977 \approx 84\text{ (ft)}}$$

e) How much time does it take the ball to reach a height of $y = 100\text{ (ft)}$? Given what we already know from part (d), we expect that there are ***no times*** when $y = 100\text{ (ft)}$. Using the quadratic formula

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)100}}{2(16.1)} = \boxed{2.2839 \pm 0.9976\, i} \quad \Rightarrow \text{no real roots}$$

f) The quadratic function $y(x)$ may be found by solving the first of Eqs. (1) for t and ***substituting*** that expression into the second of Eqs. (1) as follows

$$\boxed{y(x) = 73.54\left(\frac{x}{61.71}\right) - 16.1\left(\frac{x}{61.71}\right)^2 = 1.1917\, x - \left(4.2278\times 10^{-3}\right)x^2}$$

With this function, we can answer questions about the ball's position without involving the time t. For example, we can ask the question: At what x-values will $y = 3\text{ (ft)}$? Again using the quadratic formula, for $\boxed{\left(4.2278\times 10^{-3}\right)x^2 - 1.1917\, x + 3 = 0}$ we find

$$x_{1,2} = \frac{1.1917 \pm \sqrt{1.1917^2 - (4\times 4.2278\times 10^{-3}\times 3)}}{2\times 4.2278\times 10^{-3}} = 140.94 \pm 138.40 = \begin{cases} 2.54\text{ (ft)} \\ 279.3\text{ (ft)} \end{cases}$$

These results are the same as we found in Eq. (3).

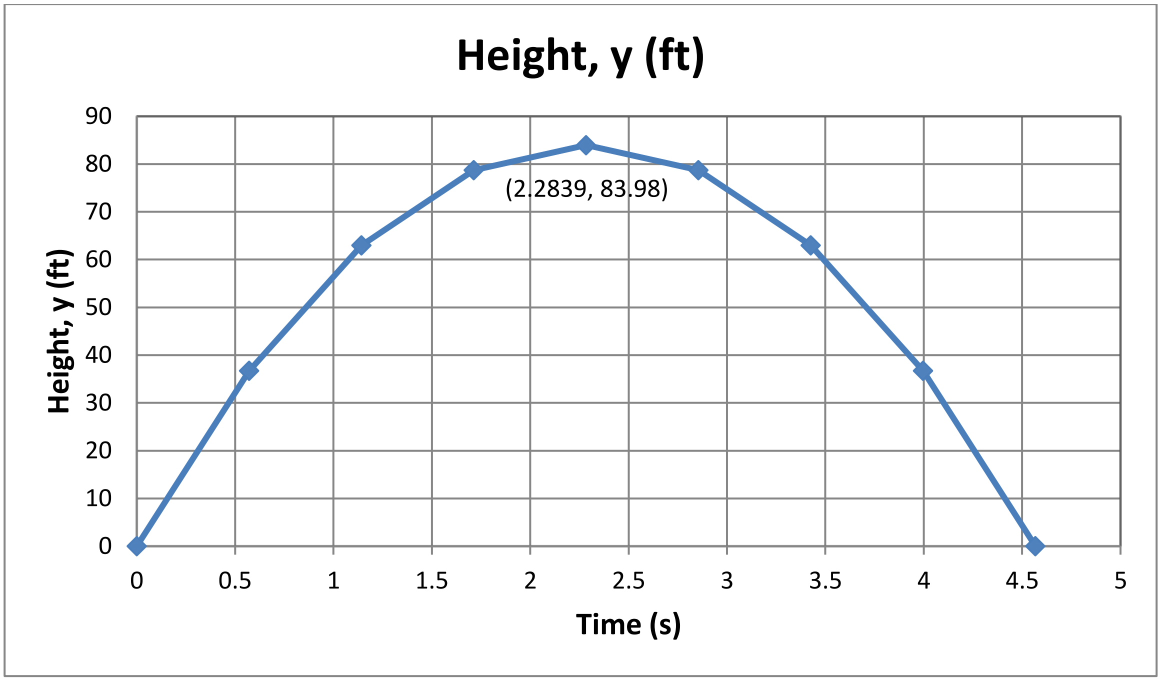


Figure 1. Height as a Function of Time

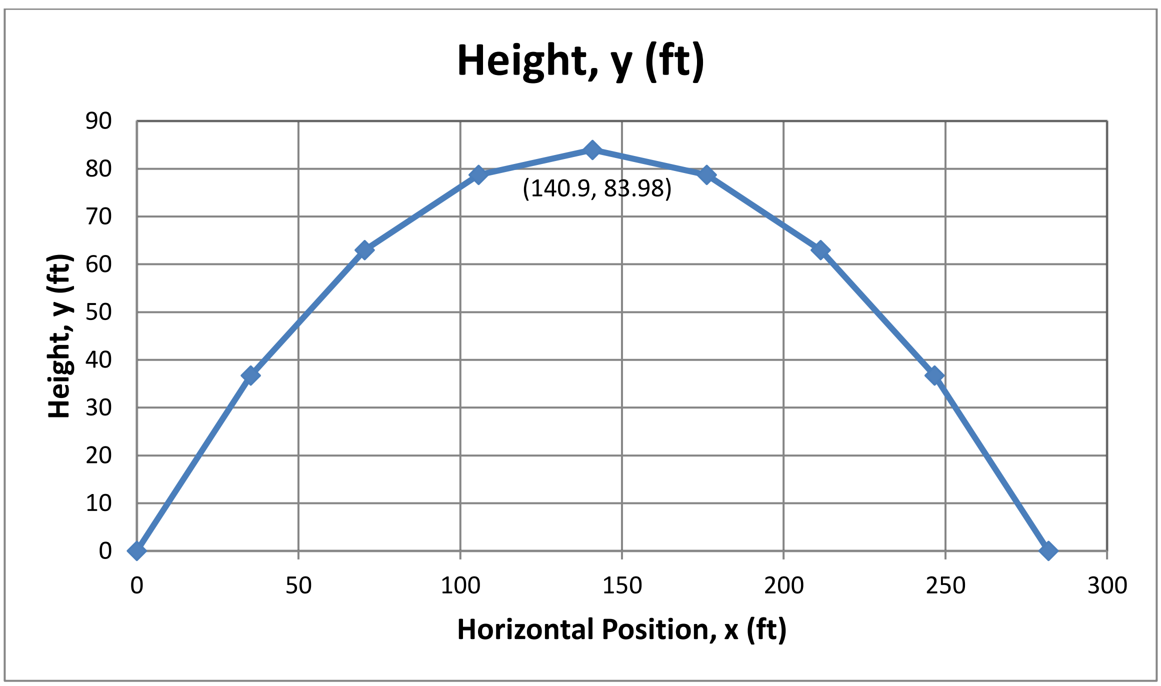


Figure 2. Height as a Function of Horizontal Position