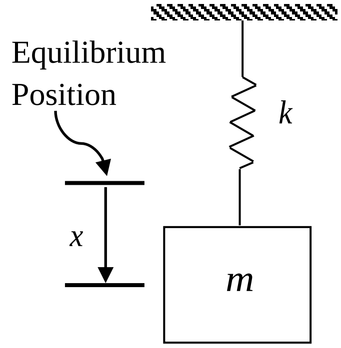
ENGR 1990 Engineering Mathematics
Equations Sheet #4 – Spring-Mass-Damper Systems

1. Spring-Mass System (no damping)



Initial Conditions: $x(0) = x_0$ $v(0) = v_0$

Natural Frequency: $\omega = \sqrt{\frac{k}{m}}$

Displacement: $x(t) = \frac{v_0}{\omega}\sin\left(\omega\, t\right) + x_0 \cos\left(\omega\, t\right)$

Velocity: $v(t) = v_0 \cos\left(\omega t\right) - x_0 \omega \sin\left(\omega t\right)$

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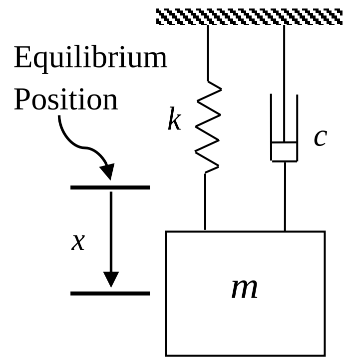
2. Spring-Mass-Damper System



Critical Damping Coefficient: $c_c = 2m\sqrt{k/m}$

Over-damped Motion: $c > c_c$

$$x(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$
$$v(t) = A \lambda_1 e^{\lambda_1 t} + B \lambda_2 e^{\lambda_2 t}$$

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} -\left(\frac{c}{2m}\right) + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{Bmatrix}$$

Coefficients: (A, B) found by solving the simultaneous equations:
$$\begin{aligned} A + B &= x_0 \\ \lambda_1 A + \lambda_2 B &= v_0 \end{aligned}$$

Under-Damped Motion: $c < c_c$

Frequency: $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

Displacement: $x(t) = e^{-\left(c/2m\right)t}\left(A\sin(\omega_d\, t) + B\cos(\omega_d\, t)\right)$

$$A = \left[v_0 + \left(\frac{c}{2m}\right)x_0\right]\Big/\omega_d$$
$$B = x_0$$

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3. Decay/Growth Rate

$$\alpha = \frac{\ln(x_2/x_1)}{t_2 - t_1}$$