ENGR 1990 Engineering Mathematics
The Integral of a Function as a Function

Previously, we learned how to estimate the integral of a function $f(x)$ over some interval $a \le x \le b$ by adding the areas of a finite set of trapezoids that represent the area under $f(x)$ from a to b. We now consider the concept of ***antiderivatives*** and how they are used to ***calculate*** the integral of a function. This idea also exposes the direct link between the concepts of ***differentiation*** and ***integration***.

Antiderivatives

A function $G(x)$ is called an ***antiderivative*** of function $f(x)$ over some interval $a \le x \le b$ if and only if $G(x)$ is continuous and $G'(x) = f(x)$ on the interval. The following table gives antiderivatives of some common functions used in engineering.

Name	Function, $f(x)$	Antiderivative, $G(x)$ $\left(G'(x) = f(x)\right)$
Constant	a	$a\,x$
Polynomial terms	$a\,x^n$	$a\,x^{n+1}/(n+1)$
Exponential	e^{ax}	e^{ax}/a
Sine	$\sin(a\,x)$	$-\cos(a\,x)/a$
Cosine	$\cos(a\,x)$	$\sin(a\,x)/a$

Note that the antiderivative is ***not unique***, because we can add a constant to any known antiderivative to produce another antiderivative. Recall that the derivative of a constant is zero.

Fundamental Theorem of Integral Calculus

If $f(x)$ is a continuous function over some interval $a \le x \le b$, and $G(x)$ is an antiderivative of $f(x)$ on that same interval, then

$$\boxed{\int_a^b f(x)dx = \int_a^b G'(x)dx = G(x)\Big|_a^b = G(b) - G(a)} \qquad (1)$$

So, if we know the function and an antiderivative, then we can calculate the integral directly. We do not have to approximate it by summing areas.

Integrals of Functions as Functions

The result of Eq. (1) is a ***number*** that represents the area under function $f(x)$ from $x = a$ to $x = b$. These results can be made more general to apply an ***arbitrary ending point*** of *x* by simply replacing the upper limit of the integral with the variable *x*.

$$\boxed{\int_a^x f(x)dx = \int_a^x G'(x)dx = G(x)\Big|_a^x = G(x) - G(a)} \tag{2}$$

This result may be used to calculate the integral for a ***starting point*** of $x = a$ to any ***ending point*** *x*.

Indefinite Integrals

When we do not specify the interval over which the integral is to be evaluated (that is, we do not specify the limits of the integral), we call the integral ***indefinite***. Effectively, we are using ***arbitrary upper*** and ***arbitrary lower*** limits. We write,

$$\boxed{\int f(x)dx = G(x) + D} \tag{3}$$

Here, D is an ***arbitrary constant***. If $G(x)$ is an antiderivative of $f(x)$ on the interval $a \le x \le b$, we can evaluate the result of Eq. (3) over that interval. As expected, we get the same result as in Eq. (1).

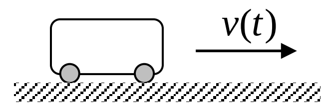
$$\boxed{\int_a^b f(x)dx = \big(G(x) + D\big)\Big|_a^b = \big(G(b) + D\big) - \big(G(a) + D\big) = G(b) - G(a)} \tag{4}$$

If we use Eq. (3) and evaluate over an arbitrary upper limit, we get the same result as in Eq. (2).

$$\boxed{\int_a^x f(x)dx = \big(G(x) + D\big)\Big|_a^x = \big(G(x) + D\big) - \big(G(a) + D\big) = G(x) - G(a)}$$

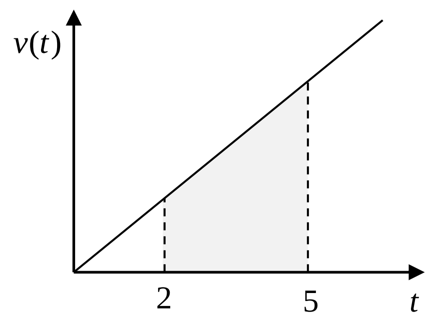
Example 1:

Given: The displacement of a car as it moves with velocity $v(t)$ from time t_1 to t_2 is the integral of $v(t)$ over that period of time.



$$\boxed{s = \int_{t_1}^{t_2} v(t)dt}$$

The displacement can be positive or negative depending on whether $v(t)$ is positive or negative.



Find: Assuming the car has velocity $v(t) = 7.5t\ (\text{ft/s}^2)$,
(a) find the displacement of the car from 2 to 5 seconds;
(b) find the total distance traveled from 2 to 5 seconds.

Solution:

(a) An antiderivative of $v(t) = 7.5t$ is $\boxed{G(t) = 7.5t^2/2 = 3.75t^2}$, so

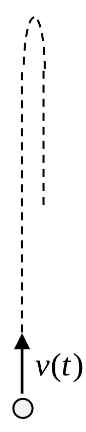
$$\boxed{s = \int_2^5 v(t)dt = \int_2^5 \left(7.5t\right)dt = 3.75t^2\Big|_2^5 = 3.75\left(5^2 - 2^2\right) = 78.75\ (\text{ft})}$$

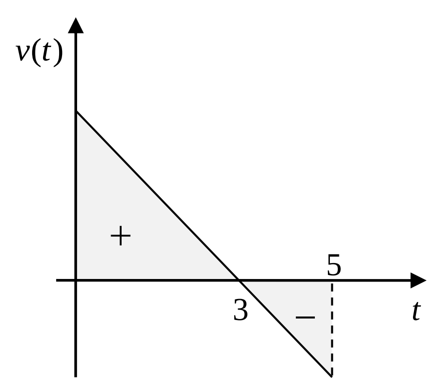
(b) As before, because $v(t)$ is positive in the range $2 \le t \le 5$, the total distance traveled is $d = s = 78.75\ (\text{ft})$.

Example 2:

Given: The velocity of a ball for a certain period of time after it is thrown upward is

$$\boxed{v(t) = 96.6 - 32.2t\ (\text{ft/s})}$$





Find: (a) the vertical displacement of the ball from 0 to 5 seconds; and (b) the total distance traveled by the ball from 0 to 5 seconds.

Solution:

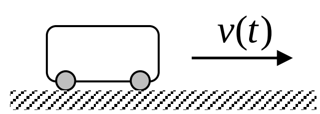
(a) An antiderivative of $v(t) = 96.6 - 32.2t$ is $G(t) = 96.6t - \left(32.2t^2/2\right) = 96.6t - 16.1t^2$, so

$$s = \int_0^5 (96.6 - 32.2t)\,dt = \left(96.6t - 16.1t^2\right)\Big|_0^5 = \left((96.6\times 5) - \left(16.1\times 5^2\right)\right) = 80.5 \text{ (ft)}$$

(b) To find the total distance traveled, we must to break the interval into two segments: $0 \le t \le 3$, and $3 \le t \le 5$.

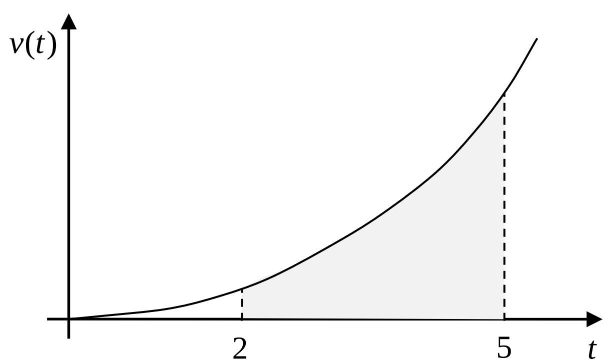
$$\begin{aligned} s &= \left|\int_0^3 (96.6 - 32.2t)\,dt\right| + \left|\int_3^5 (96.6 - 32.2t)\,dt\right| = \left|\left(96.6t - 16.1t^2\right)\Big|_0^3\right| + \left|\left(96.6t - 16.1t^2\right)\Big|_3^5\right| \\ &= \left|(96.6\times 3) - \left(16.1\times 3^2\right)\right| + \left|\left((96.6\times 5) - \left(16.1\times 5^2\right)\right) - \left((96.6\times 3) - \left(16.1\times 3^2\right)\right)\right| \\ &= |144.9| + |80.5 - 144.9| \\ &= 209.3 \text{ (ft)} \end{aligned}$$

Example 3:



Given: The velocity of a car over the time interval from 0 to 5 seconds is $v(t) = 5t^2$ (ft/s).

Find: The distance traveled by the car from 2 to 5 seconds.



Solution: Since the function is positive throughout the entire range of *t*, the total distance traveled is equal to the displacement. We can calculate the displacement by noting that $G(t) = \frac{5}{3}t^3$ is an antiderivative of $v(t) = 5t^2$. So,

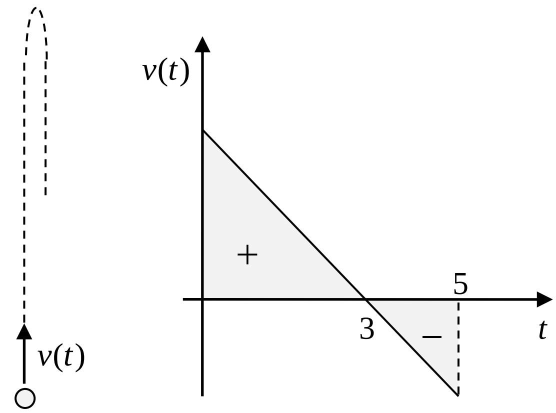
$$d = s = \int_2^5 \left(5t^2\right)dt = \frac{5}{3}\,t^3\Big|_2^5 = \frac{5}{3}\left(5^3 - 2^3\right) = \frac{5}{3}(125 - 8) = 195 \text{ (ft)}$$

Example 4:

Given: The velocity of a ball for a certain period of time after it is thrown upward is

$$\boxed{v(t) = 96.6 - 32.2t \text{ (ft/s)}}$$

Find: (a) $s(t)$, the vertical displacement of the ball as a function of time; (b) the time required for the ball to its starting point; and (c) the maximum height of the ball.



Solution:

(a) The displacement function relative to the starting point may be found by integrating to an arbitrary upper limit.

$$\boxed{s(t) = \int_0^t v(t)dt = \int_0^t (96.6 - 32.2t)dt = \left(96.6t - 16.1t^2\right)\Big|_0^t = 96.6t - 16.1t^2} \qquad (5)$$

In this result, the displacement function is zero at $t = 0$. If we want to specify a non-zero displacement at $t = 0$, say $s(0) = 50$, then we take a slightly different approach. First, we set

$$\boxed{s(t) = \int v(t)dt = 96.6t - 16.1t^2 + D}$$

Then, we solve for D to satisfy the ***initial condition***. In this case, $D = 50$.

(b) To find the time required for the ball to return to its starting point, set $s(t)$ from Eq. (5) to zero, and solve for t.

$$\boxed{s(t) = 96.6t - 16.1t^2 = (96.6 - 16.1t)t = 0} \quad \text{or} \quad \boxed{t = \begin{cases} 0 \text{ (sec)} \\ 6 \text{ (sec)} \end{cases}}$$

(c) Recall that ***maxima*** or ***minima*** of functions occur when their ***derivatives*** are ***zero***.

$$\boxed{s'(t) = \frac{d}{dt}\left(96.6t - 16.1t^2\right) = 96.6 - (16.1 \times 2)t = 96.6 - 32.2t = 0} \quad \text{or} \quad \boxed{t = 3 \text{ (sec)}}$$

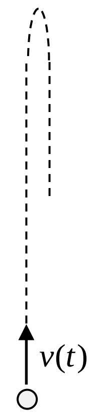
$$\boxed{s_{\max} = \left(96.6t - 16.1t^2\right)\Big|_{t=3} = \left((96.6 \times 3) - \left(16.1 \times 3^2\right)\right) = 144.9 \text{ (ft)}}$$

This corresponds physically to the condition that the ball's ***velocity*** be ***zero***.

Recall also that for the function to have a maximum, the second derivative should be ***negative***.

$$\boxed{s''(t) = \frac{d}{dt}\left(96.6 - 32.2t\right) = -32.2} \quad \text{(checks)}$$

Example 5:



Given: A ball is thrown upward with an initial velocity of 50 (ft/s) from an initial height of 6 (ft). The ball has a downward acceleration of 32.2 (ft/s^2).

$$\boxed{v(t) = \int a(t)dt} \qquad \boxed{s(t) = \int v(t)dt}$$

Find: (a) the velocity function $v(t)$; and (b) the displacement function $s(t)$.

Solution:

(a) Using the indefinite form of integration and the initial condition $v(0) = 50$ (ft/s)

$$v(t) = \int a(t)dt = \int -32.2dt = -32.2t + D \quad \Rightarrow \boxed{v(t) = 50 - 32.2t \text{ (ft/s)}}$$

(b) Using the indefinite form of integration and the initial condition $s(0) = 6$ (ft)

$$s(t) = \int v(t)dt = \int\left(50 - 32.2t\right)dt = 50t - 16.1t^2 + D \quad \Rightarrow \boxed{s(t) = 6 + 50t - 16.1t^2 \text{ (ft)}}$$