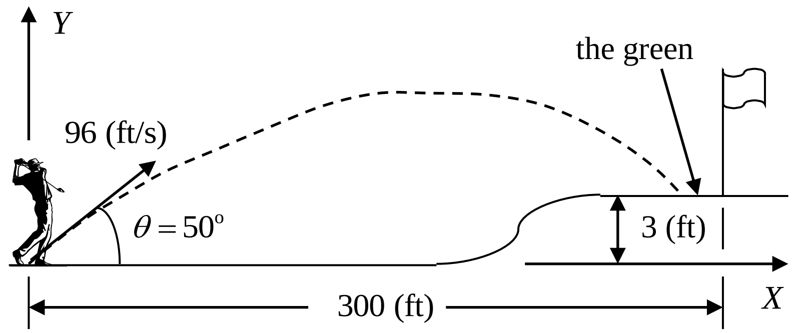
ENGR 1990 Engineering Mathematics
Introduction to Complex Numbers

Introduction



Recall that when we calculate the ***roots*** of a ***quadratic equation***, we may get real roots, or we may get a complex conjugate pair. As an example, consider the golf ball trajectory problem we discussed in earlier notes.

To find the times when the ball is ***50 feet*** above the ground $\left(y = 50\text{ (ft)}\right)$, we solved the quadratic equation $\boxed{16.1t^2 - 73.54t + 50 = 0}$ using the ***quadratic formula*** and found

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)50}}{2(16.1)} = 2.2839 \pm 1.4527 \quad \Rightarrow \boxed{t_{1,2} = \begin{cases} 0.8312 \approx 0.831\text{ (s)} \\ 3.7366 \approx 3.74\text{ (s)} \end{cases}}$$

The ball passes the 50-foot mark on its ***way up*** and on its ***way down***.

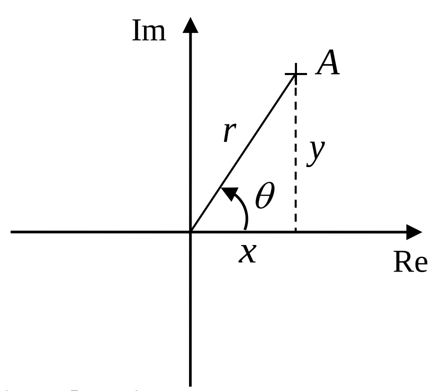
To find the times when the ball is ***100 feet*** above the ground, we solved the equation $\boxed{16.1t^2 - 73.54t + 100 = 0}$ and found

$$t_{1,2} = \frac{73.54 \pm \sqrt{73.54^2 - 4(16.1)100}}{2(16.1)} = \frac{73.54 \pm \sqrt{-1031.87}}{32.2} = \frac{73.54 \pm j\sqrt{1031.87}}{32.2}$$

$$= \frac{73.54 \pm j\,32.1227}{32.2} = \boxed{2.2839 \pm j\,0.9976}$$

The result is a ***complex conjugate pair*** $\left(j = \sqrt{-1}\right)$. This occurs because the ball ***never reaches 100 feet***, so no real solutions exist.

Complex Numbers and the Complex Plane



Generally, complex numbers have both ***real*** and ***imaginary*** parts. The diagram shows a complex number A plotted in the ***complex plane***. We can express A using either ***rectangular*** or ***polar*** coordinates.

Rectangular form: $\boxed{A = x + j\,y}$

Polar Form: $\boxed{A = r\,e^{j\theta}}$ or $\boxed{A = r\angle\theta}$

We can relate the rectangular and polar forms using ***right-triangle trigonometry***.

Given the ***rectangular form*** $\boxed{A = x + j\,y}$, we can find the ***polar form*** $\boxed{A = r\,e^{j\theta}}$.

$\boxed{r = \sqrt{x^2 + y^2} = |A|}$ …the magnitude of A

$\boxed{\theta = \tan^{-1}\left(y/x \right)}$ …the phase angle of A

Given the ***polar form*** $\boxed{A = r\,e^{j\theta}}$, we can find the ***rectangular form*** $\boxed{A = x + j\,y}$.

$\boxed{x = r\cos(\theta)}$ …the real part of A

$\boxed{y = r\sin(\theta)}$ …the imaginary part of A

From these results we can identify Euler's formula:

$$A = x + j\,y = (r\cos(\theta)) + j(r\sin(\theta)) = r(\cos(\theta) + j\sin(\theta)) = r\,e^{j\theta}$$

or

$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$ …***Euler's formula***

Complex Conjugate

Given a complex number $A = a_1 + j\,a_2$, the complex conjugate of a A is defined to be

$\boxed{A^* = a_1 - j\,a_2}$ …the complex conjugate

Operations with Complex Numbers

Addition/Subtraction

Addition and subtraction of complex numbers is most easily done in ***rectangular form***. Given two complex numbers $A = a_1 + j\,a_2$ and $B = b_1 + j\,b_2$, then

$\boxed{A + B = (a_1 + b_1) + j\,(a_2 + b_2)}$ and $\boxed{A - B = (a_1 - b_1) + j\,(a_2 - b_2)}$

If A and B are given in ***polar form***, then it is best to ***convert*** them to rectangular form before adding or subtracting.

Multiplication/Division (Polar Form)

Multiplication and division of complex numbers is most easily done in ***polar form***.

$\boxed{A \times B = \left(ae^{j\alpha}\right)\left(be^{j\beta}\right) = ab\,e^{j(\alpha+\beta)}}$ and $\boxed{A / B = \left(ae^{j\alpha}\right)/\left(be^{j\beta}\right) = (a/b)\,e^{j(\alpha-\beta)}}$

$\boxed{A \times B = (a\angle\alpha)(b\angle\beta) = ab\angle(\alpha + \beta)}$ and $\boxed{A/B = (a\angle\alpha)/(b\angle\beta) = (a/b)\angle(\alpha - \beta)}$

If A and B are given in ***rectangular form***, it is usually best to ***convert*** them to polar form before multiplying or dividing.

Multiplication/Division (Rectangular Form)

Multiplication and division of complex numbers can also be done (with a little more work) using ***rectangular form***.

Multiplication

Given two complex numbers $A = a_1 + j\,a_2$ and $B = b_1 + j\,b_2$, then their product is

$$\boxed{A \times B = (a_1 b_1 - a_2 b_2) + j\,(a_1 b_2 + a_2 b_1)} \quad \left(j \times j = -1\right)$$

Note that if *B* is the complex conjugate of *A* $\left(B = A^*\right)$, then the product is a real number equal to the square of the magnitude of *A*.

$$\boxed{A \times A^* = (a_1^2 + a_2^2) + j\,(a_1 a_2 - a_2 a_1) = a_1^2 + a_2^2 = |A|^2}$$

Division

To compute the ratio of *A* and *B* is a little more involved. To ensure that the imaginary parts appear only in the numerator, we must make use of the complex conjugate.

$$\frac{A}{B} = \frac{a_1 + j\,a_2}{b_1 + j\,b_2} = \left(\frac{a_1 + j\,a_2}{b_1 + j\,b_2}\right) \cdot \left(\frac{b_1 - j\,b_2}{b_1 - j\,b_2}\right) = \frac{(a_1 b_1 + a_2 b_2) + j(a_2 b_1 - a_1 b_2)}{b_1^2 + b_2^2}$$

Example #1

Given: $A = 5 + j\,10$ Find: the polar form of A

Solution: $\boxed{r = |A| = \sqrt{5^2 + 10^2} = \sqrt{125} \approx 11.2}$

$\boxed{\theta = \tan^{-1}(10/5) = 1.107 \text{ (rad)} = 63.4 \text{ (deg)}}$

Example #2

Given: $A = -5 + j\,10$ Find: the polar form of *A*

Solution: $\boxed{r = |A| = \sqrt{5^2 + 10^2} = \sqrt{125} \approx 11.2}$

$\boxed{\theta = \tan^{-1}(10/-5) = -1.107 + \pi = 2.03 \text{ (rad)} = 117 \text{ (deg)}}$

Example #3

Given: $A = 5 + j10$ Find: $A \times A^*$

Solution: $$\boxed{A \times A^* = (5 + j10) \times (5 - j10) = 5^2 + 10^2 = 125}$$

Example #4

Given: $A = 5 + j10$ and $B = 3 - j8$ Find: $A + B$, $A \times B$ and A/B

Solution: $$\boxed{A + B = (5 + j10) + (3 - j8) = 8 - j2}$$

$$A \times B = (5 + j10) \times (3 - j8) = ((5 \times 3) - (-8 \times 10) + j((3 \times 10) - (5 \times 8))$$

$$\boxed{A \times B = 95 - j10}$$

$$A/B = \frac{(5 + j10)}{(3 - j8)} = \frac{(5 + j10) \times (3 + j8)}{(3 - j8) \times (3 + j8)} = \frac{((15 - 80) + j(30 + 40))}{3^2 + 8^2} = \frac{-65 + j70}{73}$$

$$\boxed{A/B = -0.89 + j0.959}$$

Example #5

Given: $A = 5e^{j(\pi/3)}$ and $B = 8e^{j(-\pi/6)}$ Find: $A \times B$ and A/B

Solution: $$\boxed{A \times B = \left(5e^{j(\pi/3)}\right) \times \left(8e^{j(-\pi/6)}\right) = (5 \times 8)\, e^{j(\pi/3 + (-\pi/6))} = 40e^{j(\pi/6)}}$$

$$\boxed{A/B = \left(5e^{j(\pi/3)}\right) \Big/ \left(8e^{j(-\pi/6)}\right) = (5/8)\, e^{j(\pi/3 - (-\pi/6))} = 0.625e^{j(\pi/2)}}$$

Example #6

Given: $A = 5e^{j(\pi/3)}$ and $B = 8e^{j(-\pi/6)}$ Find: $A + B$

Solution: We must first ***convert*** the polar forms to rectangular forms, then add

$$\boxed{A = 5e^{j(\pi/3)} = 5(\cos(\pi/3) + j\sin(\pi/3)) = 2.5 + j4.33}$$

$$\boxed{B = 8e^{j(-\pi/6)} = 8(\cos(-\pi/6) + j\sin(-\pi/6)) = 6.928 - j4}$$

$$\boxed{A + B = 9.43 + j0.33}$$