ENGR 1990 Engineering Mathematics
Introduction to Differential Equations in Mechanical Engineering

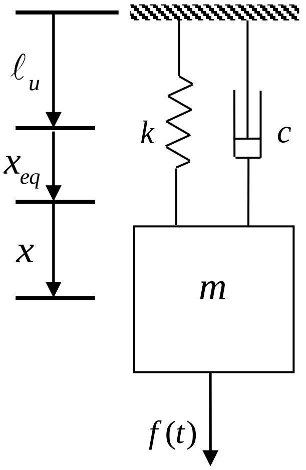
Differential equations are used to ***model*** (simulate) the response of physical systems. Their solutions allow us to ***predict*** the response of these systems without first building a prototype for testing. When used in this way, we refer to these equations as ***mathematical models***. These models can be used to simulate the response of very complex systems. The design of automobiles, airplanes, bridges, chemical plants, electronic circuits, power grids, biomedical devices, road systems (for improved traffic flow), and many others depend on them heavily.

Unfortunately, most differential equations do not have readily obtainable analytical solutions; however, they can be solved numerically using computer programs that employ a variety of numerical methods. However, as powerful as these methods are, it is often difficult to use them to draw general conclusions (and develop insight) about the many types of responses a system can exhibit. This is due to the fact that we can only observe one solution at a time.

There are many classes of differential equations, however, for which we can find analytical solutions, and these solutions often provide reasonable predictions of the response of real systems. These solutions can be useful for developing insight into the nature of a system's response. In these notes, we consider the solutions of ***second-order, ordinary differential equations*** with ***constant coefficients***.

Mass-Spring-Damper System

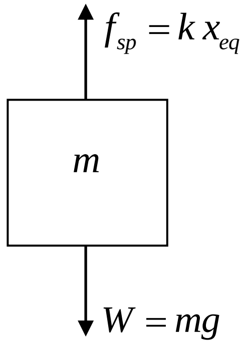
m : mass of the block
k : spring stiffness $\left(f_{\text{sp}} = -k\,x \right)$
c : coefficient of the damper $\left(f_{\text{damper}} = -c\,dx/dt = -c\,\dot{x} \right)$
$f(t)$: applied external force (input)
ℓ_u : unstretched (natural) length of spring
x_{eq} : static equilibrium position of mass
x : position of mass *relative to* the equilibrium position (output)
$\dot{x}$: $v = dx/dt$, velocity of the mass
$\ddot{x}$: $a = dv/dt = d^2x/dt^2$, acceleration of the mass



Static Equilibrium

For the mass to be in static equilibrium, the sum of the forces must be zero. We can use this fact to find the ***equilibrium position*** of the mass under its own weight.

$$\boxed{+\downarrow \sum F = mg - k\,x_{eq} = 0 \quad \Rightarrow \quad x_{eq} = mg/k} \qquad (1)$$

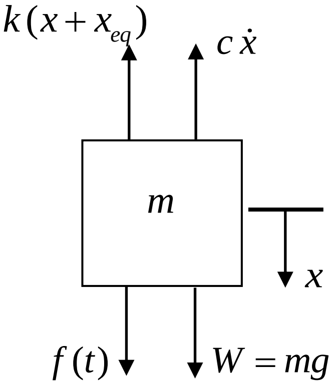


Differential Equation of Motion

Using Newton's second law, we can find the ***differential equation of motion*** of the mass. This equation can be used to ***predict*** the motion of the mass under a variety of conditions.

$$\boxed{+\downarrow \sum F = f(t) + \cancel{mg} - k\left(x + \cancel{x_{eq}}\right) - c\dot{x} = ma = m\ddot{x}}$$

$$\boxed{m\,\ddot{x} \;+\; c\,\dot{x} \;+\; k\,x = f(t)} \qquad (2)$$



Note that ***static forces are not present*** in this equation.

The solution of this equation can be used to predict the ***forced response*** of the system ***relative to the equilibrium position***. The ***free response*** of the system is described by solving the equation with $f(t) \equiv 0$. In either case, to get a ***unique solution***, we must specify the ***initial conditions***, that is, we must specify $x(0)$ the initial position of the mass and $v(0) = \dot{x}(0)$ the initial velocity of the mass.

Eq. (2) is an example of a ***second-order, linear, constant coefficient ordinary differential equation***. In the following sections, we learn how to ***solve*** these types of equations.

Solving for Free Response $\left(f(t) \equiv 0\right)$

If we set $f(t)$ to ***zero***, Eq. (2) is called a ***homogeneous differential equation***, and its solution is called the ***homogeneous solution***. We can find this solution using the following steps.

1. Substitute $x(t) = Ae^{st}$ into the differential equation and find the conditions on A and s for this to be a solution to the equation.

$$\boxed{x(t) = Ae^{st} \quad \Rightarrow \quad \dot{x}(t) = v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(Ae^{st}\right) = Ase^{st}}$$

$$\boxed{\dot{x}(t) = v(t) = Ase^{st} \quad \Rightarrow \quad \ddot{x}(t) = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(Ase^{st}\right) = As^2e^{st}}$$

So,

$$\boxed{\begin{aligned} 0 &= m\,\ddot{x} + c\,\dot{x} + k\,x = m\left(As^2e^{st}\right) + c\left(Ase^{st}\right) + k\left(Ae^{st}\right) \\ &= \left(ms^2 + cs + k\right)Ae^{st} \end{aligned}}$$

For a ***non-zero solution***, we require s to satisfy the equation $\boxed{ms^2 + cs + k = 0}$. This is called the ***characteristic equation*** of the system. There is ***no condition*** on A.

2. Find the roots of the characteristic equation and determine the ***type*** of ***response***.

$$\boxed{ms^2 + cs + k = 0 \quad \Rightarrow \quad s^2 + \left(\tfrac{c}{m}\right)s + \tfrac{k}{m} = 0}$$

Using the ***quadratic formula***,

$$\boxed{s_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}}$$

<u>Case 1</u>: real, unequal roots, $\boxed{\frac{c}{2m} > \sqrt{\frac{k}{m}}}$ (***over-damped***)

$$\boxed{s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}}$$

<u>Case 2:</u> real, equal roots, $\boxed{\frac{c}{2m} = \sqrt{\frac{k}{m}}}$ (***critical damping***)

$$\boxed{s_1 = s_2 = -\frac{c}{2m}}$$

<u>Case3:</u> complex roots, $\boxed{\frac{c}{2m} < \sqrt{\frac{k}{m}}}$ (***under-damped***)

$$\boxed{s_{1,2} = -\frac{c}{2m} \pm i\sqrt{\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2} = -\frac{c}{2m} \pm i\omega_d}$$

The frequency $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$ is the ***frequency*** in (rad/s) of the ***damped response***. If the damping is zero $\left(c = 0\right)$, then the frequency of the response is $\omega_n = \sqrt{\frac{k}{m}}$ (rad/s). This is called the system's ***natural frequency***.

The types of solutions for each of these cases are summarized in the following table.

Case	Type of Roots	Type of Motion	Form of Solution
1	Real, unequal	*Over-damped*	$x(t) = Ae^{s_1 t} + Be^{s_2 t}$
2	Real, equal	*Critically damped*	$x(t) = Ae^{st} + Bt\,e^{st}$
3	Complex conjugates	*Under-damped*	$x(t) = e^{-\left(c/2m\right)t}\left[A\sin(\omega_d\, t) + B\cos(\omega_d\, t)\right]$ $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

3. Finally, as we saw in earlier notes, the ***coefficients*** A and B are determined by applying the ***initial conditions***, $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Example 1:

Given: $m = 0.5$ (slug), $c = 15$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s)

Find: The displacement function $x(t)$.

Solution:

Using the quadratic formula, we find the roots of the characteristic equation.

$$s_{1,2} = -\tfrac{c}{2m} \pm \sqrt{\left(\tfrac{c}{2m}\right)^2 - \left(\tfrac{k}{m}\right)} = -\left(\tfrac{15}{2\times 0.5}\right) \pm \sqrt{\left(\tfrac{15}{2\times 0.5}\right)^2 - \tfrac{50}{0.5}} = \begin{Bmatrix} -3.82 \\ -26.18 \end{Bmatrix}$$

Given that we have real, unequal roots,

$$\boxed{x(\mathrm{t}) = Ae^{-3.82t} + Be^{-26.18t}}$$

Now we can apply the initial conditions:

$$\boxed{x(0) = \left(Ae^{-3.82t} + Be^{-26.18t}\right)\Big|_{t=0} = A + B = 0.25}$$

$$\boxed{\begin{aligned} \dot{x}(0) &= \left(\frac{d}{dt}\left[Ae^{-3.82t} + Be^{-26.18t}\right]\right)_{t=0} = \left(-3.82Ae^{-3.82t} - 26.18Be^{-26.18t}\right)_{t=0} \\ &= -3.82A - 26.18B \\ &= 5 \end{aligned}}$$

Or,

$$\boxed{\begin{bmatrix} 1 & 1 \\ -3.82 & -26.18 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.25 \\ 5 \end{Bmatrix}} \quad \Rightarrow \quad \boxed{\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.5163 \\ -0.2663 \end{Bmatrix}}$$

The solution is

$$\boxed{x(t)=\left((0.5163)e^{-3.82t}-(0.2663)e^{-26.18t}\right)\ \text{(ft)}}$$

Example 2:

Given: $m=0.5$ (slug), $c=5$ (lb-s/ft), $k=50$ (lb/ft), $x_0=0.25$ (ft), $v_0=5$ (ft/s)

Find: The displacement function $x(t)$.

Solution:
Using the quadratic formula, we find the roots of the characteristic equation.

$$s_{1,2}=-\tfrac{c}{2m}\pm\sqrt{\left(\tfrac{c}{2m}\right)^2-\left(\tfrac{k}{m}\right)}=-\left(\tfrac{5}{2\times0.5}\right)\pm\sqrt{\left(\tfrac{5}{2\times0.5}\right)^2-\tfrac{50}{0.5}}=-5\pm\sqrt{75}\,i$$

Given that we have ***complex roots***, the solution is of the form

$$\boxed{x(t)=e^{-(c/2m)t}\left[A\sin(\omega_d\,t)+B\cos(\omega_d\,t)\right]=e^{-5t}\left[A\sin(\sqrt{75}\,t)+B\cos(\sqrt{75}\,t)\right]}$$

Now we can apply the ***initial conditions***:

$$\boxed{x(0)=\left(e^{-(c/2m)t}\left[A\sin(\omega_d\,t)+B\cos(\omega_d\,t)\right]\right)_{t=0}=B=0.25}$$

And, using the ***product rule*** for ***differentiation***,

$$\boxed{\begin{aligned}\dot{x}(0)&=\left\{\frac{d}{dt}\left(e^{-5t}\left[A\sin(\sqrt{75}\,t)+B\cos(\sqrt{75}\,t)\right]\right)\right\}_{t=0}\\&=\left\{-5e^{-5t}\left[A\sin(\sqrt{75}\,t)+B\cos(\sqrt{75}\,t)\right]\right\}_{t=0}+\\&\quad\left\{e^{-5t}\left[\sqrt{75}A\cos(\sqrt{75}\,t)-\sqrt{75}B\sin(\sqrt{75}\,t)\right]\right\}_{t=0}\\&=-5B+\sqrt{75}A\\&=5\end{aligned}}$$

Solving these two equations gives, $A=0.7217$ and $B=0.25$. So, our under-damped solution is

$$\boxed{x(t)=e^{-5t}\left[0.7217\sin(\sqrt{75}\,t)+0.25\cos(\sqrt{75}\,t)\right]\ \text{(ft)}}$$

In previous notes, we found this could be written as a single, phase-shifted sine function.

$$\boxed{x(t)=0.7638\,e^{-5t}\sin\left((8.6603)t+0.3335\right)\ \text{(ft)}}$$

Solving for the Forced Response $\left(f(t)\neq 0\right)$

To find the forced response, we start by finding the ***particular solution***. The particular solution can sometimes be obtained using the ***method of undetermined coefficients*** provided the forcing function $f(t)$ has a simple form. The forced response is the sum of the particular solution and the solution of the homogenous equation (steps 1 and 2, only).

$$\boxed{x(t)=x_P(t)+x_H(t)}$$

The coefficients of $x_H(t)$ are found by applying the initial conditions $x(0)=x_0$ and $\dot{x}(0)=v_0$.

Forms of the particular solution for some common engineering forcing functions are given in the table below. In the method of undetermined coefficients, we substitute the general form of $x_P(t)$ back into the differential equation and solve for the unknown coefficients (referred to as B_i in the table).

	$f(t)$	Form* of $x_P(t)$
constant	a_0	Bt^n
linear	a_1t+a_0	$\left(B_1t+B_0\right)t^n$
quadratic	$a_2t^2+a_1t+a_0$	$\left(B_2t^2+B_1t+B_0\right)t^n$
exponential	$a\,e^{\beta t}$	$\left(B_1e^{\beta t}\right)t^n$
sine or cosine	$a\sin(\omega t)$ ***or*** $a\cos(\omega t)$	$\left[B_1\sin(\omega t)+B_2\cos(\omega t)\right]t^n$
exponential/ sine or cosine product	$a\,e^{\beta t}\sin(\omega t)$ ***or*** $a\,e^{\beta t}\cos(\omega t)$	$e^{\beta t}\left[B_1\sin(\omega t)+B_2\cos(\omega t)\right]t^n$

* The ***exponent*** n is the ***smallest, non-negative integer*** so that every term in $x_P(t)$ is ***different*** from every term in $x_H(t)$. That is, $n=0$ unless the same type of term appears in $x_H(t)$.

Example 3:

Given: $m = 0.5$ (slug), $c = 15$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s), and $f(t) = 30\sin(5t)$.

Find: (a) the form of the homogeneous solution $x_H(t)$; (b) the particular solution $x_P(t)$; and (c) the displacement function $x(t)$.

Solution:

(a) Setting $f(t) = 0$, our differential equation is as it was in Example 1 above. The general form of that solution is the form or our ***homogeneous solution***.

$$\boxed{x_H(\mathrm{t}) = Ae^{-3.82t} + Be^{-26.18t}}$$

(b) Given that $f(t) = 3\sin(10t)$ is not of the form of any term in our homogeneous solution, we set $n = 0$, and the ***particular solution*** has the form

$$\boxed{x_P(t) = D\sin(5t) + E\cos(5t)}$$

We find the coefficients D and E by substituting the form of $x_P(t)$ into the differential equation and equating coefficients of like terms. First, we must differentiate $x_P(t)$.

$$\boxed{\dot{x}_P(t) = \frac{d}{dt}\big(D\sin(5t) + E\cos(5t)\big) = 5D\cos(5t) - 5E\sin(5t)}$$

$$\boxed{\ddot{x}_P(t) = \frac{d}{dt}\big(5D\cos(5t) - 5E\sin(5t)\big) = -25D\sin(5t) - 25E\cos(5t)}$$

Substituting into the differential equation: $\boxed{m\ddot{x} + c\dot{x} + kx = f(t)}$

$$\begin{aligned}
& m\ddot{x}_P + c\dot{x}_P + kx_P \\
& = m\big(-25D\sin(5t) - 25E\cos(5t)\big) + c\big(5D\cos(5t) - 5E\sin(5t)\big) \\
& +k\big(D\sin(5t) + E\cos(5t)\big) \\
& = \big(-25mD - 5cE + kD\big)\sin(5t) + \big(-25mE + 5cD + kE\big)\cos(10t) \\
& = 30\sin(5t)
\end{aligned}$$

Or,

$$\begin{bmatrix} (k-25m) & -5c \\ 5c & (k-25m) \end{bmatrix}\begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{bmatrix} 37.5 & -75 \\ 75 & 37.5 \end{bmatrix}\begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix} \quad \Rightarrow \boxed{\begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{Bmatrix} 0.16 \\ -0.32 \end{Bmatrix}}$$

So, the particular solution is

$$\boxed{x_P(t) = 0.16\sin(5t) - 0.32\cos(5t)}$$

(c) The solution is the ***sum*** of the ***homogeneous*** and ***particular solutions***.

$$\boxed{x(t) = x_H(t) + x_P(t) = Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t)}$$

To find the coefficients A and B, we ***apply*** the ***initial conditions***.

$$\boxed{x(0) = \left[Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t) \right]_{t=0} = A + B - 0.32 = 0.25}$$

$$\boxed{\begin{aligned} \dot{x}(0) &= \left(\frac{d}{dt} \left[Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t) \right] \right)_{t=0} \\ &= \left(-3.82Ae^{-3.82t} - 26.18Be^{-26.18t} + (5 \times 0.16)\cos(5t) + (5 \times 0.32)\sin(5t) \right)_{t=0} \\ &= -3.82A - 26.18B + 0.8 \\ &= 5 \end{aligned}}$$

Or,

$$\begin{bmatrix} 1 & 1 \\ -3.82 & -26.18 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.57 \\ 4.2 \end{Bmatrix} \quad \Rightarrow \quad \boxed{\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.855 \\ -0.2852 \end{Bmatrix}}$$

So, the ***solution*** is

$$\boxed{x(t) = x_H(t) + x_P(t) = \underbrace{0.855e^{-3.82t} - 0.2852e^{-26.18t}}_{\text{transient}} + \underbrace{0.16\sin(5t) - 0.32\cos(5t)}_{\text{steady-state}}}$$

It has both ***transient*** and ***steady-state*** terms. Transient terms go to ***zero*** as time progresses, but the steady-state terms do not.