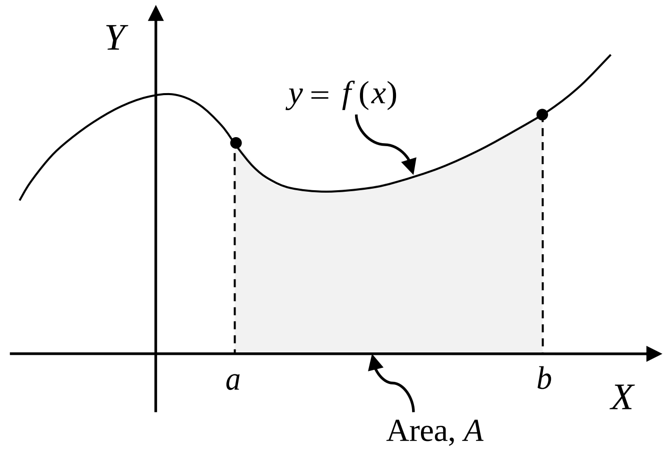
ENGR 1990 Engineering Mathematics
The Integral of a Function – An Introduction

Definite Integral

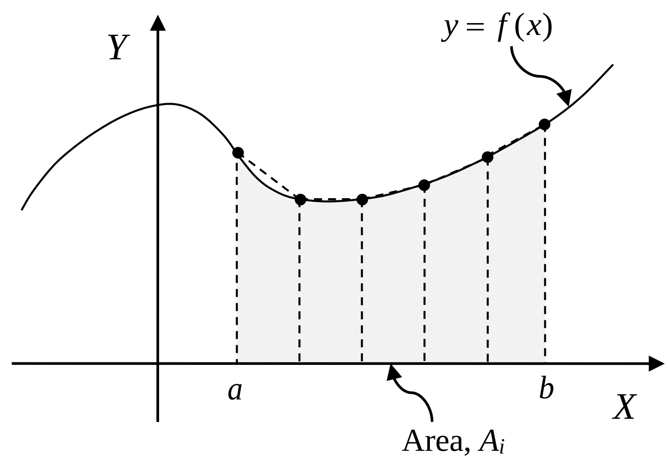
Consider a continuous function $y = f(x)$. The ***integral*** of $f(x)$ from $x = a$ to $x = b$ is simply the ***area under the curve*** between those two points. The area is bounded on the underside by the *X*-axis. We write



$$\boxed{A = \int_a^b f(x)dx} \qquad (1)$$

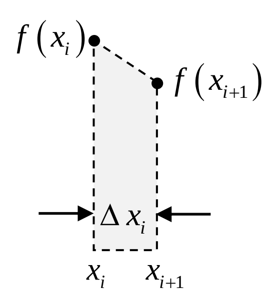
In this form, the integral is called a ***definite integral***. It is a ***number***, not a function. Later, we will define an ***indefinite integral*** which is itself a ***function*** of *x*.

One way to ***estimate*** the definite integral of a function is to break down the area into a finite number of ***trapezoids*** and ***sum*** the areas of all the trapezoids.



$$\begin{aligned} A &\approx \sum_i \left[\frac{f(x_{i+1}) + f(x_i)}{2}\right]\Delta x_i \\ &\approx \sum_i \left[\frac{f(x_{i+1}) + f(x_i)}{2}\right](x_{i+1} - x_i) \end{aligned} \qquad (2)$$

As the increments Δx_i become smaller, Eq. (2) yields a more accurate estimate of the area *A*.



So far, we considered the function $f(x)$ and the increments Δx_i to be ***positive***. Consequently, the area *A* is ***positive***. If, however, $f(x)$ is ***negative*** over this range the area will be ***negative***. Clearly, if $f(x)$ takes on both ***positive*** and ***negative*** values, the result could be ***positive***, ***negative*** or ***zero***.

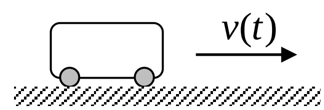
Given our current understanding of definite integrals, the following basic properties of integrals should seem reasonable.

	Property	Comment
1	$\int_a^b -f(x)dx = -\int_a^b f(x)dx$	function values have opposite sign, so, areas will also
2	$\int_b^a f(x)dx = -\int_a^b f(x)dx$	increments have opposite sign, so, areas will also
3	$\int_a^a f(x)dx = 0$	width of area = zero
4	$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$	total area = the sum of the areas
5	$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$	α is a constant
6	$\int_a^b \left(f(x) + g(x) \right) dx = \int_a^b f(x)dx + \int_a^b g(x)dx$	integral of a sum = the sum of the integrals

Note on units: The units of an integral are the same as the units of $f(x) \times \Delta x$.

Example 1:

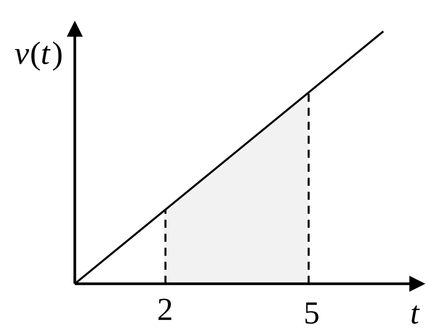
Given: The displacement of a car as it moves with velocity $v(t)$ from time t_1 to t_2 is the integral of $v(t)$ over that period of time.



$$s = \int_{t_1}^{t_2} v(t)dt$$

The displacement can be positive or negative depending on whether $v(t)$ is positive or negative.

Find: Assuming the car has velocity $v(t) = 7.5t$ (ft/s^2),
(a) find the displacement of the car from 2 to 5 seconds;
(b) find the total distance traveled from 2 to 5 seconds.



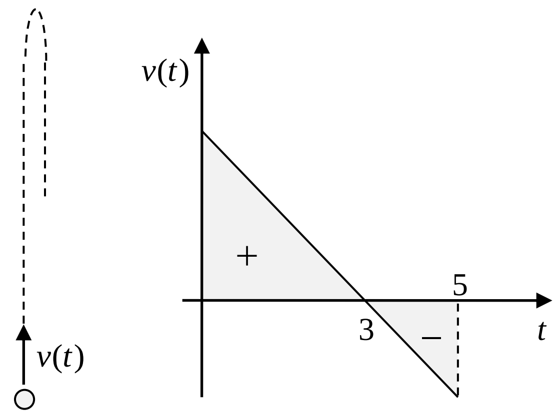
Solution:

(a) To calculate the shaded area, we can use a single trapezoid. The units of the result are the same as the units of $v(t)\times\Delta t \;\rightarrow\; (\text{ft/s})\times\text{s} \;\rightarrow\; (\text{ft})$.

$$\boxed{s = A = (5-2)\big(v(2)+v(5)\big)/2 = 3(15+37.5)/2 = 78.75 \text{ (ft)}}$$

(b) The total distance traveled is also 78.75 (ft), because the velocity of the car is positive from 2 to 5 seconds.

Example 2:



Given: The velocity of a ball for a certain period of time after it is thrown upward is

$$\boxed{v(t) = 96.6 - 32.2t \text{ (ft/s)}}$$

Find: (a) the vertical displacement of the ball from 0 to 5 seconds; and (b) the total distance traveled by the ball from 0 to 5 seconds.

Solution:

(a) the vertical displacement of the ball from 0 to 5 seconds is

$$\boxed{s = \tfrac{1}{2}(3)(96.6) - \tfrac{1}{2}(5-3)(64.4) = 80.5 \text{ (ft)}}$$

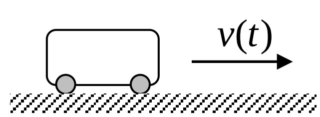
The downward movement of the ball is subtracted from the upward movement.

(b) the total distance traveled by the ball from 0 to 5 seconds is

$$\boxed{d = \tfrac{1}{2}(3)(96.6) + \tfrac{1}{2}(5-3)(64.4) = 209.3 \text{ (ft)}}$$

The upward and downward movements of the ball are summed.

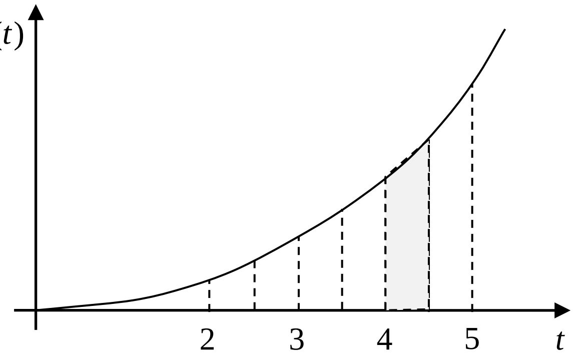
Example 3:



Given: The velocity of a car over the time interval from 0 to 5 seconds is $\boxed{v(t) = 5t^2 \text{ (ft/s)}}$.

Find: estimate the distance traveled by the car from 2 to 5 seconds.

Solution: Since the function is positive throughout the entire range of t, the total distance traveled is equal to the displacement. We can estimate the displacement by breaking up the area as shown in the diagram.



Time, t	$f(t)$	Interval	f_{avg}	Δt
2	20	--		0.5
2.5	31.25	1	25.625	0.5
3	45	2	38.125	0.5
3.5	61.25	3	53.125	0.5
4	80	4	70.625	0.5
4.5	101.25	5	90.625	0.5
5	125	6	113.125	0.5
		Σ	391.25	

Since $\Delta t = 0.5\ \text{(s)}$ for all intervals, we have

$$s = A \approx \sum_{i=1}^{6}\left[\frac{f\left(x_{i+1}\right)+f\left(x_i\right)}{2}\right]\Delta t_i = \sum_{i=1}^{6}\left(f_{\text{avg}}\right)_i \Delta t_i = \Delta t \sum_{i=1}^{6}\left(f_{\text{avg}}\right)_i \approx 0.5 \times 391.25 \approx 195.625\ \text{(ft)}$$

We will see later that the actual displacement is 195 (ft). This is an error of about 0.32%.