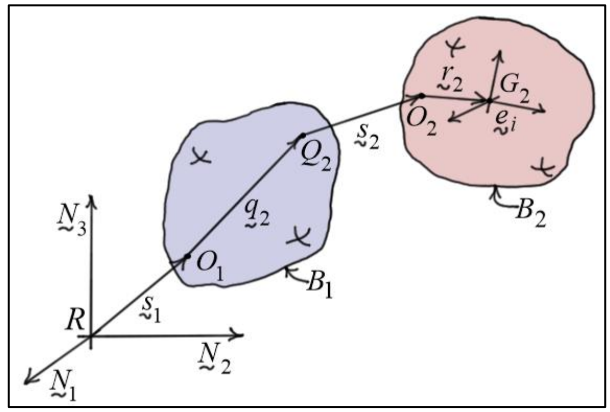
ME 6590 Multibody Dynamics
Accelerations Using Relative Coordinates

<u>Velocities</u>

Consider again the ***two-body system*** shown. In previous notes, it was found that the ***fixed-frame components*** of ${}^{R}\underset{\sim}{v}_{G_2}$ the ***velocity*** of G_2 in the fixed frame R can be written as



$$\left\{v_{G_2}\right\} = \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\dot{s}_2'\right\} - \left[\tilde{r}_2\right]\left[R_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\} \qquad (1)$$

or

$$\begin{aligned}\left\{v_{G_2}\right\} = \left\{\dot{s}_1\right\} - \left[R_{B_1}\right]^T \left(\left[\tilde{q}_2'\right] + \left[\tilde{s}_2'\right]\right)\left\{\omega_{B_1}'\right\} - \left[R_{B_2}\right]^T \left[\tilde{r}_2'\right]\left[{}^{B_2}R_{B_1}\right]^T \left\{\omega_{B_1}'\right\} \\ + \left[R_{B_1}\right]^T \left\{\dot{s}_2'\right\} - \left[R_{B_2}\right]^T \left[\tilde{r}_2'\right]\left\{\hat{\omega}_{B_2}'\right\}\end{aligned} \qquad (2)$$

The first equation is written in terms of the ***fixed-frame components*** of ${}^{R}\underset{\sim}{\omega}_{B_1}$ and the B_1 ***frame components*** of ${}^{B_1}\underset{\sim}{\omega}_{B_2}$, and the second is written in terms of the ***body-frame components*** of the angular velocities.

<u>Accelerations</u>

<u>Case 1: Fixed-Frame Angular Velocity Components as Generalized Speeds</u>

If the generalized speeds are defined to be the elements of $\left\{\dot{s}_1\right\}$, $\left\{\dot{s}_2'\right\}$, $\left\{\omega_{B_1}\right\}$, and $\left\{\hat{\omega}_{B_2}\right\}$, then write

$$\boxed{\left\{v_{G_2}\right\} = \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\} + \left[v_{G_2,\dot{s}_2'}\right]\left\{\dot{s}_2'\right\} + \left[v_{G_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}} \qquad (3)$$

where

$\left[v_{G_2,\dot{s}_1}\right]$ is a 3×3 identity matrix

$\left[v_{G_2,\omega_{B_1}}\right] = -\left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)$

$$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2,\hat{\omega}_{B_2}}\right] = -\left[\tilde{r}_2\right]\left[R_{B_1}\right]^T$$

Differentiating Eq. (3), the ***fixed-frame components*** of the ***acceleration*** of G_2 can then be written as

$$\boxed{\begin{aligned}\left\{a_{G_2}\right\} &= \left[v_{G_2,\dot{s}_1}\right]\left\{\ddot{s}_1\right\} + \left[v_{G_2,\omega_{B_1}}\right]\left\{\dot{\omega}_{B_1}\right\} + \left[\dot{v}_{G_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\} \\ &\quad + \left[v_{G_2,\dot{s}'_2}\right]\left\{\ddot{s}'_2\right\} + \left[\dot{v}_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\hat{\omega}_{B_2}}\right]\left\{\dot{\hat{\omega}}_{B_2}\right\} + \left[\dot{v}_{G_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}\end{aligned}} \qquad (4)$$

where the time derivatives of the partial velocity matrices can be written as follows.

a) $\left[\dot{v}_{G_2,\omega_{B_1}}\right] = -\left(\left[\dot{\tilde{q}}_2\right] + \left[\dot{\tilde{s}}_2\right] + \left[\dot{\tilde{r}}_2\right]\right)$

The elements of the three matrices on the right side can be calculated as

$$\left\{\dot{q}_2\right\} = \frac{d}{dt}\left(\left[R_{B_1}\right]^T\left\{q'_2\right\}\right) = \left[\dot{R}_{B_1}\right]^T\left\{q'_2\right\} = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\left\{q'_2\right\} \qquad (5)$$

$$\begin{aligned}\left\{\dot{s}_2\right\} &= \frac{d}{dt}\left(\left[R_{B_1}\right]^T\left\{s'_2\right\}\right) = \left[\dot{R}_{B_1}\right]^T\left\{s'_2\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} \\ &= \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\left\{s'_2\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\}\end{aligned} \qquad (6)$$

$$\left\{\dot{r}_2\right\} = \frac{d}{dt}\left(\left[R_{B_2}\right]^T\left\{r'_2\right\}\right) = \left[\dot{R}_{B_2}\right]^T\left\{r'_2\right\} = \left[\tilde{\omega}_{B_2}\right]\left[R_{B_2}\right]^T\left\{r'_2\right\} \qquad (7)$$

b) $\left[\dot{v}_{G_2,\dot{s}'_2}\right] = \left[\dot{R}_{B_1}\right]^T = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T$ (8)

c) $\left[\dot{v}_{G_2,\hat{\omega}_{B_2}}\right] = -\left[\dot{\tilde{r}}_2\right]\left[R_{B_1}\right]^T - \left[\tilde{r}_2\right]\left[\dot{R}_{B_1}\right]^T = -\left[\dot{\tilde{r}}_2\right]\left[R_{B_1}\right]^T - \left[\tilde{r}_2\right]\left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T$ (9)

The elements of $\left[\dot{\tilde{r}}_2\right]$ are calculated in Eq. (7) above.

Case 2: Body-Frame Angular Velocity Components as Generalized Speeds

If the ***generalized speeds*** are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\{\omega'_{B_1}\}$, and $\{\hat{\omega}'_{B_2}\}$, then

$$\boxed{\{v_{G_2}\} = [v_{G_2,\dot{s}_1}]\{\dot{s}_1\} + [v_{G_2,\omega'_{B_1}}]\{\omega'_{B_1}\} + [v_{G_2,\dot{s}'_2}]\{\dot{s}'_2\} + [v_{G_2,\hat{\omega}'_{B_2}}]\{\hat{\omega}'_{B_2}\}} \qquad (10)$$

where

$[v_{G_2,\dot{s}_1}]$ is a 3×3 identity matrix

$$[v_{G_2,\omega'_{B_1}}] = -[R_{B_1}]^T([\tilde{q}'_2]+[\tilde{s}'_2]) - [R_{B_2}]^T[\tilde{r}'_2][{}^{B_2}R_{B_1}]^T$$

$$[v_{G_2,\dot{s}'_2}] = [R_{B_1}]^T$$

$$[v_{G_2,\hat{\omega}'_{B_2}}] = -[R_{B_2}]^T[\tilde{r}'_2]$$

Differentiating Eq. (10), the ***fixed-frame components*** of the ***acceleration*** of G_2 can then be written as

$$\boxed{\begin{aligned}\{a_{G_2}\} &= [v_{G_2,\dot{s}_1}]\{\ddot{s}_1\} + [v_{G_2,\omega'_{B_1}}]\{\dot{\omega}'_{B_1}\} + [\dot{v}_{G_2,\omega'_{B_1}}]\{\omega'_{B_1}\} \\ &\quad + [v_{G_2,\dot{s}'_2}]\{\ddot{s}'_2\} + [\dot{v}_{G_2,\dot{s}'_2}]\{\dot{s}'_2\} + [v_{G_2,\hat{\omega}'_{B_2}}]\{\dot{\hat{\omega}}'_{B_2}\} + [\dot{v}_{G_2,\hat{\omega}'_{B_2}}]\{\hat{\omega}'_{B_2}\}\end{aligned}}$$

where the time derivatives of the partial velocity matrices can be written as follows.

a) $$\begin{aligned}[\dot{v}_{G_2,\omega'_{B_1}}] &= -[R_{B_1}]^T[\dot{\tilde{s}}'_2] - [\dot{R}_{B_1}]^T([\tilde{q}'_2]+[\tilde{s}'_2]) - [\dot{R}_{B_2}]^T[\tilde{r}'_2][{}^{B_2}R_{B_1}]^T \\ &\quad - [R_{B_2}]^T[\tilde{r}'_2][{}^{B_1}\dot{R}_{B_2}] \\ &= -[R_{B_1}]^T[\dot{\tilde{s}}'_2] - [R_{B_1}]^T[\tilde{\omega}'_{B_1}]([\tilde{q}'_2]+[\tilde{s}'_2]) \\ &\quad - [R_{B_2}]^T[\tilde{\omega}'_{B_2}][\tilde{r}'_2][{}^{B_2}R_{B_1}]^T - [R_{B_2}]^T[\tilde{r}'_2][\tilde{\hat{\omega}}'_{B_2}][{}^{B_1}R_{B_2}]\end{aligned}$$

b) $$[\dot{v}_{G_2,\dot{s}'_2}] = [\dot{R}_{B_1}]^T = [R_{B_1}]^T[\tilde{\omega}'_{B_1}]$$

c) $$[\dot{v}_{G_2,\hat{\omega}'_{B_2}}] = -[\dot{R}_{B_2}]^T[\tilde{r}'_2] = -[R_{B_2}]^T[\tilde{\omega}'_{B_2}][\tilde{r}'_2]$$

Case 3: Orientation Angle Derivatives as Generalized Speeds

If, instead, the generalized speeds are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}_2\}$, and a set of ***orientation angle derivatives***, then $\{\omega_{B_1}\}$ and $\{\hat{\omega}_{B_2}\}$ (or $\{\omega'_{B_1}\}$ and $\{\hat{\omega}'_{B_2}\}$) can be written in terms of the orientation angles. Returning to Eq. (1), for example,

$$\begin{aligned}\{v_{G_2}\} &= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)\{\omega_{B_1}\} + [R_{B_1}]^T \{\dot{s}'_2\} - [\tilde{r}_2][R_{B_1}]^T \{\hat{\omega}_{B_2}\} \\ &= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)[\omega_{B_1,\dot{\theta}_{B_1}}]\{\dot{\theta}_{B_1}\} + [R_{B_1}]^T \{\dot{s}'_2\} - [\tilde{r}_2][\omega_{B_2,\dot{\theta}_{B_2}}]\{\dot{\theta}_{B_2}\} \\ &= [v_{G_2,\dot{s}_1}]\{\dot{s}_1\} + [v_{G_2,\dot{\theta}_{B_1}}]\{\dot{\theta}_{B_1}\} + [v_{G_2,\dot{s}'_2}]\{\dot{s}'_2\} + [v_{G_2,\dot{\theta}_{B_2}}]\{\dot{\theta}_{B_2}\}\end{aligned} \qquad (11)$$

where

$[v_{G_2,\dot{s}_1}]$ is a 3×3 identity matrix

$$[v_{G_2,\dot{\theta}_{B_1}}] = -\left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)[\omega_{B_1,\dot{\theta}_{B_1}}]$$

$$[v_{G_2,\dot{s}'_2}] = [R_{B_1}]^T$$

$$[v_{G_2,\dot{\theta}_{B_2}}] = -[\tilde{r}_2][\omega_{B_2,\dot{\theta}_{B_2}}]$$

The partial angular velocity matrices are defined in previous notes for a 1-2-3 rotation sequence.

The fixed-frame components of the acceleration of G_2 can then be written as

$$\begin{aligned}\{a_{G_2}\} &= [v_{G_2,\dot{s}_1}]\{\ddot{s}_1\} + [v_{G_2,\dot{\theta}_{B_1}}]\{\ddot{\theta}_{B_1}\} + [\dot{v}_{G_2,\dot{\theta}_{B_1}}]\{\dot{\theta}_{B_1}\} + [v_{G_2,\dot{s}'_2}]\{\ddot{s}'_2\} + [\dot{v}_{G_2,\dot{s}'_2}]\{\dot{s}'_2\} \\ &\quad + [v_{G_2,\dot{\theta}_{B_2}}]\{\ddot{\theta}_{B_2}\} + [\dot{v}_{G_2,\dot{\theta}_{B_2}}]\{\dot{\theta}_{B_2}\}\end{aligned} \qquad (12)$$

where

$$[\dot{v}_{G_2,\dot{\theta}_{B_1}}] = -\left([\dot{\tilde{q}}_2] + [\dot{\tilde{s}}_2] + [\dot{\tilde{r}}_2]\right)[\omega_{B_1,\dot{\theta}_{B_1}}] - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2]\right)[\dot{\omega}_{B_1,\dot{\theta}_{B_1}}]$$

$$[\dot{v}_{G_2,\dot{s}'_2}] = [\dot{R}_{B_1}]^T = [\tilde{\omega}_{B_1}][R_{B_1}]^T$$

$$[\dot{v}_{G_2,\dot{\theta}_{B_2}}] = -[\dot{\tilde{r}}_2][\omega_{B_2,\dot{\theta}_{B_2}}] - [\tilde{r}_2][\dot{\omega}_{B_2,\dot{\theta}_{B_2}}]$$