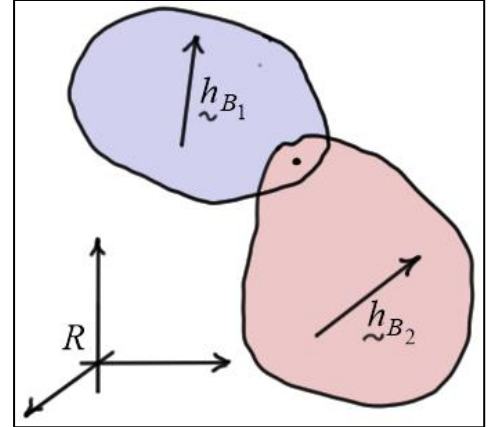


ME 6590 Multibody Dynamics: Connecting Joints – Part II

Two-Angle (Universal) Joint: Absolute Coordinates

- A *universal joint* is one that allows two bodies to share a *common point* (requiring three translation constraints), and it allows them to have *two relative rotational degrees of freedom* (requiring a single constraint on rotation).
- The three *constraints on translation* are the *same* as those given for a *spherical joint*.
- To formulate the *constraint on rotation*, consider the two bodies shown in the figure. Let $\boxed{\tilde{h}_{B_1}}$ be a vector fixed in B_1 that describes the *direction* of the *first rotation*, and let $\boxed{\tilde{h}_{B_2}}$ be a vector fixed in B_2 that describes the *direction* of the *second rotation*.
- The *constraint* can then be written



$$\boxed{(\tilde{h}_{B_1} \times \tilde{h}_{B_2}) \cdot {}^{B_1}\omega_{B_2} = 0} \quad \text{or} \quad \boxed{(\tilde{h}_{B_2} \times \tilde{h}_{B_1}) \cdot {}^{B_1}\omega_{B_2} = 0} \quad (1)$$

- Using *inertial components* of the *angular velocities*, we have

$$\begin{aligned} 0 &= \left([\tilde{h}_{B_1}] [R_{B_2}]^T \{h'_{B_2}\} \right)^T \left(\{\omega_{B_2}\} - \{\omega_{B_1}\} \right) = \{h'_{B_2}\}^T [R_{B_2}] [\tilde{h}_{B_1}]^T \left(\{\omega_{B_2}\} - \{\omega_{B_1}\} \right) \\ &= \{h'_{B_2}\}^T [R_{B_2}] [\tilde{h}_{B_1}] \left(\{\omega_{B_1}\} - \{\omega_{B_2}\} \right) \end{aligned}$$

The last equation can be *differentiated* to give

$$\begin{aligned} 0 &= \{h'_{B_2}\}^T [R_{B_2}] [\tilde{h}_{B_1}] \left(\{\dot{\omega}_{B_1}\} - \{\dot{\omega}_{B_2}\} \right) \\ &\quad + \{h'_{B_2}\}^T \left([R_{B_2}] [\dot{\tilde{h}}_{B_1}] + [\dot{R}_{B_2}] [\tilde{h}_{B_1}] \right) \left(\{\omega_{B_1}\} - \{\omega_{B_2}\} \right) \end{aligned}$$

or

$$\boxed{\begin{aligned} 0 &= \{h'_{B_2}\}^T [R_{B_2}] [\tilde{h}_{B_1}] \left(\{\dot{\omega}_{B_1}\} - \{\dot{\omega}_{B_2}\} \right) \\ &\quad + \{h'_{B_2}\}^T \left([R_{B_2}] [\dot{\tilde{h}}_{B_1}] - [R_{B_2}] [\dot{\omega}_{B_2}] [\tilde{h}_{B_1}] \right) \left(\{\omega_{B_1}\} - \{\omega_{B_2}\} \right) \end{aligned}} \quad (2)$$

The elements of $\left[\begin{smallmatrix} \tilde{h}_{B_1} \end{smallmatrix} \right]$ are found by using the vector components of

$$\left\{ \dot{h}_{B_1} \right\} = \left[\tilde{\omega}_{B_1} \right] \left[R_{B_1} \right]^T \left\{ h'_{B_1} \right\}$$

- Using **body-fixed components** of the **angular velocities**, it is more convenient to write the constraint equation as

$$\begin{aligned} 0 &= \left(\underline{h}_{B_1} \times \underline{h}_{B_2} \right) \cdot \left(\underline{\omega}_{B_2} - \underline{\omega}_{B_1} \right) = \left(\underline{h}_{B_1} \times \underline{h}_{B_2} \right) \cdot \underline{\omega}_{B_2} - \left(\underline{h}_{B_1} \times \underline{h}_{B_2} \right) \cdot \underline{\omega}_{B_1} \\ &= \left(\underline{h}_{B_1} \times \underline{h}_{B_2} \right) \cdot \underline{\omega}_{B_2} + \left(\underline{h}_{B_2} \times \underline{h}_{B_1} \right) \cdot \underline{\omega}_{B_1} \\ &= \underline{h}_{B_1} \cdot \left(\underline{h}_{B_2} \times \underline{\omega}_{B_2} \right) + \underline{h}_{B_2} \cdot \left(\underline{h}_{B_1} \times \underline{\omega}_{B_1} \right) \end{aligned}$$

or, in matrix form,

$$\boxed{\{0\} = \left\{ h'_{B_1} \right\}^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{h}'_{B_2} \right] \left\{ \omega'_{B_2} \right\} + \left\{ h'_{B_2} \right\}^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{h}'_{B_1} \right] \left\{ \omega'_{B_1} \right\}} \quad (3)$$

- This result can be **differentiated** to put the constraint in terms of the angular acceleration components.

$$\begin{aligned} \{0\} &= \left\{ h'_{B_1} \right\}^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{h}'_{B_2} \right] \left\{ \dot{\omega}'_{B_2} \right\} + \left\{ h'_{B_2} \right\}^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{h}'_{B_1} \right] \left\{ \dot{\omega}'_{B_1} \right\} \\ &\quad + \left\{ h'_{B_1} \right\}^T \left(\left[\dot{R}_{B_1} \right] \left[R_{B_2} \right]^T + \left[R_{B_1} \right] \left[\dot{R}_{B_2} \right]^T \right) \left[\tilde{h}'_{B_2} \right] \left\{ \omega'_{B_2} \right\} \\ &\quad + \left\{ h'_{B_2} \right\}^T \left(\left[\dot{R}_{B_2} \right] \left[R_{B_1} \right]^T + \left[R_{B_2} \right] \left[\dot{R}_{B_1} \right]^T \right) \left[\tilde{h}'_{B_1} \right] \left\{ \omega'_{B_1} \right\} \\ &= \left\{ h'_{B_1} \right\}^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{h}'_{B_2} \right] \left\{ \dot{\omega}'_{B_2} \right\} + \left\{ h'_{B_2} \right\}^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{h}'_{B_1} \right] \left\{ \dot{\omega}'_{B_1} \right\} \\ &\quad + \left\{ h'_{B_1} \right\}^T \left(\left[\tilde{\omega}'_{B_1} \right]^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T + \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{\omega}'_{B_2} \right] \right) \left[\tilde{h}'_{B_2} \right] \left\{ \omega'_{B_2} \right\} \\ &\quad + \left\{ h'_{B_2} \right\}^T \left(\left[\tilde{\omega}'_{B_2} \right]^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T + \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{\omega}'_{B_1} \right] \right) \left[\tilde{h}'_{B_1} \right] \left\{ \omega'_{B_1} \right\} \end{aligned} \quad (4)$$

or

$$\boxed{\begin{aligned} \{0\} &= \left\{ h'_{B_1} \right\}^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{h}'_{B_2} \right] \left\{ \dot{\omega}'_{B_2} \right\} + \left\{ h'_{B_2} \right\}^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{h}'_{B_1} \right] \left\{ \dot{\omega}'_{B_1} \right\} \\ &\quad + \left\{ h'_{B_1} \right\}^T \left(\left(\left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{\omega}'_{B_1} \right] \right)^T + \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{\omega}'_{B_2} \right] \right) \left[\tilde{h}'_{B_2} \right] \left\{ \omega'_{B_2} \right\} \\ &\quad + \left\{ h'_{B_2} \right\}^T \left(\left(\left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{\omega}'_{B_2} \right] \right)^T + \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{\omega}'_{B_1} \right] \right) \left[\tilde{h}'_{B_1} \right] \left\{ \omega'_{B_1} \right\} \end{aligned}} \quad (5)$$

Two-Angle (Universal) Joint: Relative Coordinates

- Using *relative coordinates* and B_1 *components* of ${}^{B_1}\omega_{B_2}$, the constraint equation

$$0 = \left(\underline{h}_{B_2} \times \underline{h}_{B_1} \right) \cdot {}^{B_1}\omega_{B_2} = \underline{h}_{B_2} \cdot \left(\underline{h}_{B_1} \times {}^{B_1}\omega_{B_2} \right) \quad (6)$$

or, in matrix-vector form

$$\left\{ \underline{h}'_{B_2} \right\}^T \left[R_{B_2} \right] \left[R_{B_1} \right]^T \left[\tilde{\underline{h}}_{B_1} \right] \left\{ \hat{\omega}_{B_2} \right\} = \{0\} \quad (7)$$

- Using the B_2 *components* of ${}^{B_1}\omega_{B_2}$, the constraint equation can be written as

$$0 = \left(\underline{h}_{B_1} \times \underline{h}_{B_2} \right) \cdot {}^{B_1}\omega_{B_2} = \underline{h}_{B_1} \cdot \left(\underline{h}_{B_2} \times {}^{B_1}\omega_{B_2} \right)$$

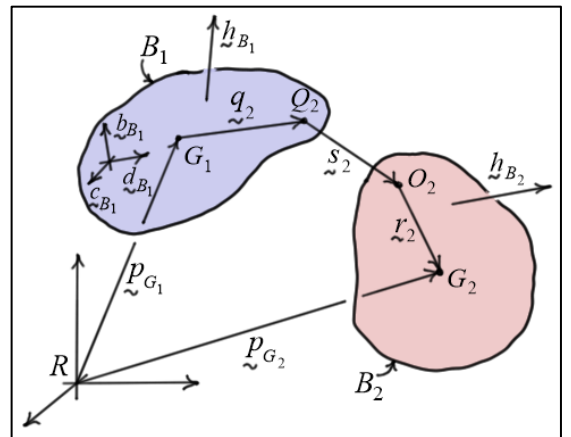
or, in matrix-vector form

$$\left\{ \underline{h}'_{B_1} \right\}^T \left[R_{B_1} \right] \left[R_{B_2} \right]^T \left[\tilde{\underline{h}}_{B_2} \right] \left\{ \hat{\omega}'_{B_2} \right\} = \{0\} \quad (8)$$

- Eqs. (7) and (8) can be *differentiated* to express the constraints in terms of the angular acceleration components.

Two-Angle Joint with One Translational Degree-of-Freedom

- The *rotational constraints* for this joint are the *same as* for the *universal joint* described above.
- Given only one translational degree of freedom, *two constraints of translation* must be written.
- Consider the *two bodies* shown. Let \underline{d}_{B_1} be in the *direction* of the *displacement* of O_2 relative to Q_2 , and let \underline{b}_{B_1} and \underline{c}_{B_1} be *perpendicular* to \underline{d}_{B_1} .



- The *constraints on translation* can then be written as follows

$$\underline{b}_{B_1} \cdot \underline{s}_2 = 0 \quad \underline{c}_{B_1} \cdot \underline{s}_2 = 0 \quad (9)$$

- Using *absolute coordinates*, the constraint equations can be written

$$\underline{b}_{B_1} \cdot (\underline{p}_{G_2} - \underline{p}_{G_1} - \underline{q}_2 - \underline{r}_2) = 0$$

$$\underline{c}_{B_1} \cdot (\underline{p}_{G_2} - \underline{p}_{G_1} - \underline{q}_2 - \underline{r}_2) = 0$$

Or, in matrix-vector form

$$\left\{ \underline{b}'_{B_1} \right\}^T \left[R_{B_1} \right] \left(\left\{ p_{G_2} \right\} - \left\{ p_{G_1} \right\} - \left[R_{B_1} \right]^T \left\{ q'_2 \right\} - \left[R_{B_2} \right]^T \left\{ r'_2 \right\} \right) = 0 \quad (10)$$

$$\left\{ \underline{c}'_{B_1} \right\}^T \left[R_{B_1} \right] \left(\left\{ p_{G_2} \right\} - \left\{ p_{G_1} \right\} - \left[R_{B_1} \right]^T \left\{ q'_2 \right\} - \left[R_{B_2} \right]^T \left\{ r'_2 \right\} \right) = 0 \quad (11)$$

- Using **relative coordinates**, the matrix-vector forms of the constraint equations are

$$\left\{ \underline{b}'_{B_1} \right\}^T \left\{ \underline{s}'_2 \right\} = 0 \quad (12)$$

$$\left\{ \underline{c}'_{B_1} \right\}^T \left\{ \underline{s}'_2 \right\} = 0 \quad (13)$$

- **Note:** Eqs. (10)-(13) can be differentiated to put them into the form of second order differential equations.