ME 6590 Multibody Dynamics
Constraint Relaxation Method: Meaning of Lagrange Multipliers

- Previously, it was noted that if a dynamic system is described using "*n*" ***generalized coordinates*** $q_k \; (k=1,\ldots,n)$, and if the system is subjected to "*m*" ***independent holonomic*** and/or ***nonholonomic*** constraint equations of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m) \tag{1}$$

  then the equations of motion of the system can be found by using one of the following two forms of Lagrange's equations with Lagrange multipliers.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{2}$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{3}$$

Here,

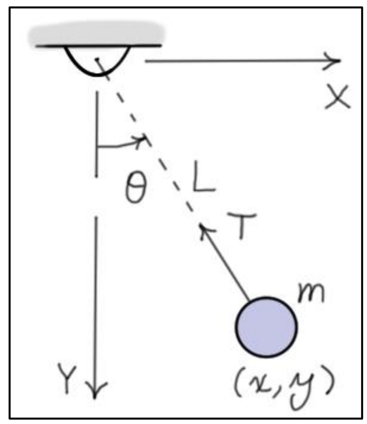
- *K* is the ***kinetic energy*** of the system.
- F_{q_k} is the ***generalized force*** associated with the generalized coordinate q_k.
- *L* is the ***Lagrangian*** of the system.
- *V* is the ***potential energy*** function for the ***conservative*** forces and torques.
- $\left(F_{q_k}\right)_{nc}$ is the ***generalized force*** associated with q_k for ***only*** the ***nonconservative*** forces and torques.
- λ_j is the Lagrange multiplier associated with the j^{th} constraint equation.
- $a_{jk} \; (j=1,\ldots,m; \; k=1,\ldots,n)$ are the coefficients from the constraint equations.

- Eqs. (1) and Eqs. (2) or (3) form a set of "$n+m$" ***differential/algebraic equations*** for the "*n*" ***generalized coordinates*** and the "*m*" ***Lagrange multipliers***.

- Alternatively, some or all the constraints can be ***relaxed*** (or ***removed***) and replaced with ***force*** and/or ***torque*** components that are required to ***maintain*** the ***constraints***. Then, formulate the "*n*" Lagrange's equations in terms of the "*n*" generalized coordinates and the "*m*" constraint force (or torque) components.
- ***Together*** with the ***constraint equations***, this forms a set of "$n+m$" ***differential/ algebraic equations*** for the "*n*" ***generalized coordinates*** and the "*m*" ***constraint force*** and/or ***torque components***. If ***all*** the constraints are ***relaxed***, then Eqs. (3) can be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \left(F_{q_k}\right)_{\text{constraints}}} \qquad (k=1,\ldots,n) \tag{4}$$

Example: The Simple Pendulum



- For the simple pendulum shown, $q_1 = x$ and $q_2 = y$ are used as the generalized coordinates, and the ***length constraint*** of the pendulum is ***relaxed*** in the formulation. Lagrange's equations can then be written in the form of Eq. (4).
- Here, $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy$, $\left(F_{q_k}\right)_{nc} = 0$, and the contributions of the constraint force to the right sides of the equations are

$$\boxed{\left(F_x\right)_{\text{constraint}} = \underset{\sim}{T} \cdot \left(\partial \underset{\sim}{v} / \partial \dot{x}\right) = T\left(-\left(x/L\right)\underset{\sim}{i} - \left(y/L\right)\underset{\sim}{j}\right) \cdot \partial\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \Big/ \partial \dot{x} = -T\left(x/L\right)} \tag{5}$$

$$\boxed{\left(F_y\right)_{\text{constraint}} = \underset{\sim}{T} \cdot \left(\partial \underset{\sim}{v} / \partial \dot{y}\right) = T\left(-\left(x/L\right)\underset{\sim}{i} - \left(y/L\right)\underset{\sim}{j}\right) \cdot \partial\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \Big/ \partial \dot{y} = -T\left(y/L\right)} \tag{6}$$

- Substituting into Lagrange's equations (4) and supplementing with the twice differentiated constraint equation gives the following equations of motion.

$$\boxed{\begin{aligned} m\ddot{x} + \left(\tfrac{x}{L}\right)T &= 0 \\ m\ddot{y} - mg + \left(\tfrac{y}{L}\right)T &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \tag{7}$$

- Using ***Lagrange multipliers***, it was shown in previous notes that the equations for the pendulum could be written as

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x = 0 \\ m\ddot{y} - mg - \lambda y = 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0 \end{aligned}} \tag{8}$$

- Comparing Eqs. (7) and (8), it is clear that the ***Lagrange multiplier*** $\lambda = -T/L$, the tension force per unit pendulum length.