

ME 6590 Multibody Dynamics

Constraints for Multibody Systems

Configuration (or Holonomic) Constraints

- Suppose the configuration of a multibody system is defined by “ n ” generalized coordinates, say q_k ($k=1, \dots, n$).
- These coordinates may all be *independent*, or they may form a *dependent set*.
- For example, consider the simple pendulum. The coordinate set $\{x_G, y_G, \theta\}$ is a *dependent set*. The coordinates may be related as follows

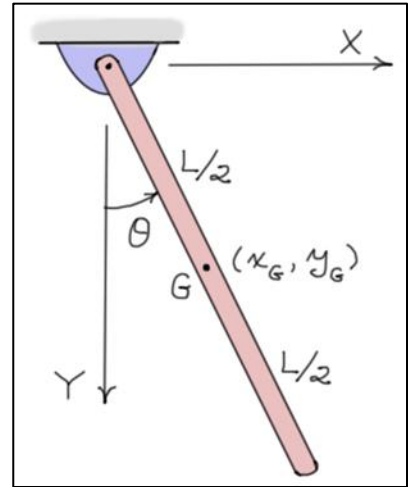
$$x_G = \frac{L}{2} \sin(\theta) \quad y_G = \frac{L}{2} \cos(\theta).$$

Hence, for this system only *one* generalized coordinate is required. Any set of *two or more* coordinates forms a *dependent set*.

- The types of constraints described above are referred to as *configuration constraints*. For a mechanical system described by “ n ” *generalized coordinates* q_k ($k=1, \dots, n$) with “ m ” *configuration constraints*, the constraints may be written as

$$\boxed{f_j(q_1, q_2, \dots, q_n, t) = 0} \quad (j=1, \dots, m) \quad (1)$$

- Here, the constraints *may also be dependent on the time, t* .
- Configuration constraints are called *holonomic* constraints. If they *depend on the time t* , they are *rheonomic* constraints. If they *do not depend on the time t* , they are called *schleronomic* constraints.
- Configuration constraints are most useful in a dynamic analysis when they are *differentiated* into a form that is *linear* in the *time derivatives* of the *generalized coordinates* or *generalized speeds*.
- Differentiating Eq. (1) gives



$$\boxed{\frac{df_j}{dt} = \sum_{k=1}^n \left(\frac{\partial f_j}{\partial q_k} \right) \dot{q}_k + \frac{\partial f_j}{\partial t} = \sum_{k=1}^n a_{jk} \dot{q}_k + a_{j0} = 0} \quad (j = 1, \dots, m) \quad (2)$$

Examples of Configuration Constraints

1. For the simple pendulum shown above, define the generalized coordinate set as

$\{q_1, q_2, q_3\} = \{x_G, y_G, \theta\}$. The two configuration constraints are

$$\begin{cases} f_1(q_1, q_2, q_3) \\ f_2(q_1, q_2, q_3) \end{cases} = \begin{cases} q_1 - \frac{L}{2} \sin(q_3) \\ q_2 - \frac{L}{2} \cos(q_3) \end{cases} = \begin{cases} 0 \\ 0 \end{cases}.$$

These equations may be differentiated to get

$$\begin{cases} \dot{q}_1 - \frac{L}{2} \dot{q}_3 \cos(q_3) \\ \dot{q}_2 + \frac{L}{2} \dot{q}_3 \sin(q_3) \end{cases} = \begin{bmatrix} 1 & 0 & -\frac{L}{2} \cos(q_3) \\ 0 & 1 & +\frac{L}{2} \sin(q_3) \end{bmatrix} \begin{cases} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{cases} = \begin{cases} 0 \\ 0 \end{cases}$$

Comparing these results to the general form shown in Eq. (2), we conclude

$$\boxed{a_{11} = a_{22} = 1} \quad \boxed{a_{12} = a_{21} = 0} \quad \boxed{a_{13} = -\frac{L}{2} \cos(\theta)} \quad \boxed{a_{23} = +\frac{L}{2} \sin(\theta)}$$

and

$$\boxed{a_{j0} = 0} \quad (j = 1, 2)$$

2. For the simple pendulum shown above, define the generalized coordinate set as

$\{q_1, q_2\} = \{x_G, y_G\}$. The configuration constraint is

$$f(q_1, q_2) = q_1^2 + q_2^2 - (L/2)^2 = 0$$

and differentiating gives

$$2q_1 \dot{q}_1 + 2q_2 \dot{q}_2 = 0 \quad \text{or} \quad \boxed{[q_1 \quad q_2] \begin{cases} \dot{q}_1 \\ \dot{q}_2 \end{cases} = 0}$$

So, in this case, $\boxed{a_{11} = q_1 = x_G}$, $\boxed{a_{12} = q_2 = y_G}$, and $\boxed{a_{10} = 0}$.

Motion Constraints

- *Motion constraints* will occur directly in the *form of* Eq. (2). It *may* or *may not* be possible to *integrate* the *constraint* to find a configuration form of the constraint as given by Eq. (1). Motion constraints that *cannot* be *integrated* are called *nonholonomic constraints*.
- It may be *difficult to tell*, however, whether a motion constraint is *integrable* or not, so it may be *difficult* to *determine* if a constraint is *holonomic* or *nonholonomic*.
- For example, suppose the motion constraint is of the form of Eq. (2). If the constraint is *holonomic*, then

$$a_{jk} = \frac{\partial f_j}{\partial q_k}$$

Moreover, if $f_j(q_1, q_2, \dots, q_n, t)$ and all its derivatives are continuous,

$$\frac{\partial a_{jk}}{\partial q_\ell} = \frac{\partial^2 f_j}{\partial q_\ell \partial q_k} = \frac{\partial^2 f_j}{\partial q_k \partial q_\ell} = \frac{\partial a_{j\ell}}{\partial q_k} \quad k, \ell = (1, \dots, n) \quad (3)$$

and

$$\frac{\partial a_{j0}}{\partial q_\ell} = \frac{\partial^2 f_j}{\partial q_\ell \partial t} = \frac{\partial^2 f_j}{\partial t \partial q_\ell} = \frac{\partial a_{j\ell}}{\partial t} \quad \ell = (1, \dots, n) \quad (4)$$

- If, for a given motion constraint, Eqs. (3) and (4) are *satisfied*, then the motion constraint is *holonomic*. However, if these equations are *not true*, we *cannot conclude* that the constraint is *nonholonomic*, because it is possible that an *integrating factor* $g_j(q_1, q_2, \dots, q_n, t)$ exists that transforms the equation into an *exact differential*.
- If an integrating factor exists, then

$$\frac{\partial(g_j a_{jk})}{\partial q_\ell} = \frac{\partial(g_j a_{j\ell})}{\partial q_k} \quad k, \ell = (1, \dots, n) \quad \frac{\partial(g_j a_{j0})}{\partial q_\ell} = \frac{\partial(g_j a_{j\ell})}{\partial t} \quad \ell = (1, \dots, n) \quad (5)$$

- Unfortunately, *finding the integrating factor is not always an easy task*.