

ME 6590 Multibody Dynamics

Generalized Speeds, Partial Angular Velocities, and Partial Velocities

Generalized Speeds

- To define the *configuration* of a multibody system with “ n ” degrees of freedom, we require at least “ n ” *generalized coordinates*, say q_s ($s=1,\dots,n$). *Generalized speeds* for the system can be defined as \dot{q}_s ($s=1,\dots,n$), the time derivatives of the generalized coordinates, or as linear combinations of the time derivatives.

- Defining the “ n ” generalized speeds as linear combinations of the \dot{q}_s ($s=1,\dots,n$), write

$$u_r = \sum_{s=1}^n (Y_{rs} \dot{q}_s) + z_r \quad (r=1,\dots,n) \quad (1)$$

Or, in matrix notation

$$\{u\} = [Y]\{\dot{q}\} + \{z\} \quad (2)$$

- Here, the elements of the matrix $[Y]$ and the vector $\{z\}$ can be *functions* of the *generalized coordinates*, q_s ($s=1,\dots,n$) *and time*, t .

- In order for q_s ($s=1,\dots,n$) and u_r ($r=1,\dots,n$) to *completely* describe the configuration of the system, the matrix $[Y]$ must be *non-singular*. Under these circumstances, Eq. (2) can be solved for $\{\dot{q}\}$ in terms of $\{u\}$ to get

$$\{\dot{q}\} = [Y]^{-1} (\{u\} - \{z\}) = [W]\{u\} + \{x\} \quad (3)$$

- Here, $[W] = [Y]^{-1}$ and $\{x\} = [Y]^{-1} \{z\}$. Eq. (3) represents a set of “ n ” *first-order, kinematical differential equations*.

- *One example* of generalized speeds are the *angular velocity components* of a rigid body. In earlier notes, the body-fixed components of the angular velocity of a rigid body were written in terms of 1-2-3 body-fixed sequence of rotation angles as

$$\{\omega'\} = \begin{bmatrix} C_2 C_3 & S_3 & 0 \\ -C_2 S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix} \{\dot{\theta}\} = [Y]\{\dot{\theta}\} + \{z\} \quad (4)$$

where $[Y]$ is the coefficient matrix of $\{\dot{\theta}\}$, and $\{z\} = \{0\}$.

- It was also found that the **body-fixed components** of the **angular velocity** of a rigid body can be written in terms of a set of four **Euler parameters** as

$$\{\omega'\} = 2[E']\{\dot{\epsilon}\} \quad (5)$$

- Eqs. (4) and (5) can both be inverted to solve for the time derivatives of the generalized coordinates to give

$$\{\dot{\theta}\} = \begin{bmatrix} (C_3/C_2) & (-S_3/C_2) & 0 \\ S_3 & C_3 & 0 \\ (-C_3S_2/C_2) & (S_2S_3/C_2) & 1 \end{bmatrix} \{\omega'\} = [W]\{\omega'\} + \{x\} \quad (6)$$

$$\{\dot{\epsilon}\} = \frac{1}{2}[E']^T \{\omega'\} = [W]\{\omega'\} + \{x\} \quad (7)$$

- In each case, $[W]$ is the **coefficient matrix** of $\{\omega'\}$, and $\{x\} = \{0\}$. Recall that the coefficient matrix of Eq. (6) is **singular** when the second orientation angle is $\pi/2$, otherwise it is **non-singular**. The coefficient matrix in Eq. (7) is **non-singular** for all body positions.

Partial Angular Velocities and Partial Velocities

- If q_r ($r=1, \dots, n$) are **independent generalized coordinates** for a **holonomic** system with “ n ” degrees of freedom, the **angular velocities** of the bodies and the **velocities** of the mass centers of the bodies can be written in the form

$$\omega_{B_k} = \sum_{r=1}^n (\omega_{B_k, \dot{q}_r} \dot{q}_r) + (\omega_{B_k})_t \quad \text{or} \quad \{\omega_{B_k}\} = [\omega_{B_k, \dot{q}}]\{\dot{q}\} + \{\omega_{B_k}\}_t \quad (8)$$

$$v_{G_k} = \sum_{r=1}^n (v_{G_k, \dot{q}_r} \dot{q}_r) + (v_{G_k})_t \quad \text{or} \quad \{v_{G_k}\} = [v_{G_k, \dot{q}}]\{\dot{q}\} + \{v_{G_k}\}_t \quad (9)$$

where ω_{B_k, \dot{q}_r} , v_{G_k, \dot{q}_r} , $(\omega_{B_k})_t$, and $(v_{G_k})_t$ can be functions of q_r ($r=1, \dots, n$) and **time**.

- The vectors ω_{B_k, \dot{q}_r} and v_{G_k, \dot{q}_r} are **partial angular velocities** and **partial velocities** of the system associated with the generalized coordinates q_r ($r=1, \dots, n$).

- If u_r ($r=1,\dots,n$) is an **independent set** of **generalized speeds** as defined above, then a set of **partial angular velocities** and **partial velocities** can be defined associated with these **generalized speeds** as well.

$$\{\omega_{B_k}\} = [\omega_{B_k,u}] \{u\} + \{\bar{\omega}_{B_k}\}_t \quad (10)$$

$$\{v_{G_k}\} = [v_{G_k,u}] \{u\} + \{\bar{v}_{G_k}\}_t \quad (11)$$

- As before, the matrices $[\omega_{B_k,u}]$, $[v_{G_k,u}]$, and the vectors $\{\bar{\omega}_{B_k}\}_t$ and $\{\bar{v}_{G_k}\}_t$ on the right side of these equations can be **functions** of the **generalized coordinates** and **time**.
- Defining a set of generalized speeds as in Eqs. (2) and (3), the partial angular velocities of Eqs. (8) and (10) can be related as follows

$$\begin{aligned} \{\omega_{B_k}\} &= [\omega_{B_k,\dot{q}}]([W]\{u\} + \{x\}) + \{\omega_{B_k}\}_t \\ &= [\omega_{B_k,\dot{q}}][W]\{u\} + ([\omega_{B_k,\dot{q}}]\{x\} + \{\omega_{B_k}\}_t) \\ &= [\omega_{B_k,u}]\{u\} + \{\bar{\omega}_{B_k}\}_t \end{aligned}$$

where

$$\boxed{[\omega_{B_k,u}] = [\omega_{B_k,\dot{q}}][W]} \quad (12)$$

$$\boxed{\{\bar{\omega}_{B_k}\}_t = [\omega_{B_k,\dot{q}}]\{x\} + \{\omega_{B_k}\}_t} \quad (13)$$

A similar result is true for the **partial velocities**.

- Note that partial velocities and partial angular velocities are usually found by **writing Eqs. (8)-(11) directly**, and then determining the partial velocities **by inspection**. Eqs. (12) and (13) show, however, that we can also **convert** one set into the other **given the kinematic relationships** of Eqs. (2) and (3).

Systems with Constraints

- As discussed in earlier notes, many **configuration** and **motion constraints** can be written as

$$\sum_{k=1}^n (a_{jk} \dot{q}_k) + a_{j0} = 0 \quad (j=1,\dots,m) \quad (14)$$

- Equivalently, these constraint equations could be expressed in terms of a set of generalized speeds as

$$\sum_{s=1}^n (b_{js} u_s) + b_{j0} = 0 \quad (j = 1, \dots, m) \quad (15)$$

- Because Eqs. (14) and (15) represent the *same* constraints, it can be shown that the coefficients a_{jk} and b_{js} ($k = 0, \dots, n$) are *related* as follows

$$b_{js} = \sum_{k=1}^n a_{jk} W_{ks} \quad (j = 1, \dots, m; s = 1, \dots, n) \quad (16)$$

$$b_{j0} = a_{j0} + \sum_{k=1}^n a_{jk} x_k \quad (j = 1, \dots, m) \quad (17)$$

- If the constraints equations (15) are *independent*, then the set of linear equations can be solved for “ m ” of the generalized speeds in terms of the remaining “ $n-m$ ” generalized speeds. Without loss of generality, assume the first “ $n-m$ ” generalized speeds form an independent set, then write

$$u_{n-m+r} = \sum_{s=1}^{n-m} b'_{rs} u_s + b'_{r0} \quad (r = 1, \dots, m) \quad (18)$$

- Given this result, all but the first “ $n-m$ ” generalized speeds can be *eliminated* from the expressions for the angular velocities and mass center velocities. Then, a new *independent* set of partial angular velocities and partial velocities can be defined associated with the “ $n-m$ ” independent generalized speeds.
- **Note:** *Eliminating dependent generalized speeds* from the angular velocities and the mass center velocities is *not as complicated* as *eliminating dependent generalized coordinates*. In fact, while the resulting expressions contain only the independent generalized speeds, *they may still contain* the *complete* (dependent) *set* of generalized coordinates.