

ME 6590 Multibody Dynamics

Moments and Products of Inertia and the Inertia Matrix

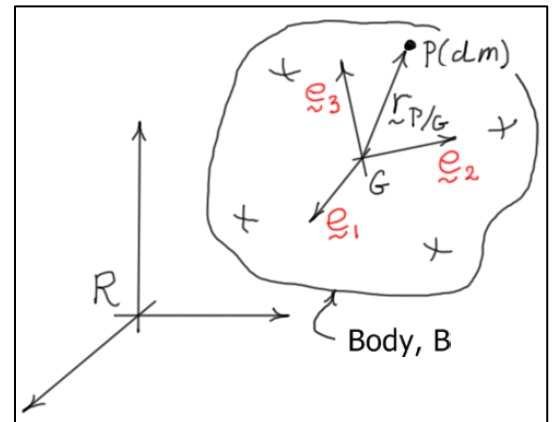
Moments of Inertia

- Consider the rigid body B . The *unit vectors* $B: (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ are *fixed* in the body and directed along a *convenient* set of axes *passing through the mass center* G .
- The *moments of inertia* of the body about these axes are defined as

$$I_{xx}^G = \int_B (y^2 + z^2) dm$$

$$I_{yy}^G = \int_B (x^2 + z^2) dm$$

$$I_{zz}^G = \int_B (x^2 + y^2) dm$$



- Here, $x, y,$ and z are defined as the \underline{e}_i components of $\underline{r}_{P/G}$ the position vector of P with respect to G , that is, $\underline{r}_{P/G} = x\underline{e}_1 + y\underline{e}_2 + z\underline{e}_3$.
- *Moments of inertia* of a body about a particular axis are a *measure* of the *distribution of the body's mass about that axis*. The *smaller the inertia* the more the mass is *concentrated* about the axis.
- Inertia values can be found either by *measurement* or *calculation*. Calculations are based on direct integration or on the "*body build-up*" technique. In the body build-up technique, inertias of *simple shapes* are *added* (or subtracted) to estimate the inertia of a composite shape.
- The inertias of *simple shapes* (about their individual mass centers) are found in *standard inertia tables*. These values are *transferred* to axes through the composite mass center using the *Parallel Axes Theorem for Moments of Inertia*.

Parallel Axes Theorem for Moments of Inertia

- The inertia (I_i^A) of a body about an axis (i) through any point (A) is equal to the inertia (I_i^G) of the body about a parallel axis through the mass center G plus the mass (m) times the distance (d_i) between the two axes squared.

$$\boxed{I_{ii}^A = I_{ii}^G + m d_i^2} \quad (i = x, y, \text{ or } z)$$

- **Moments of inertia are always positive.** From the parallel axes theorem, it is obvious that the **minimum moments of inertia** of a body occur about axes that pass through its **mass center**.

Products of Inertia

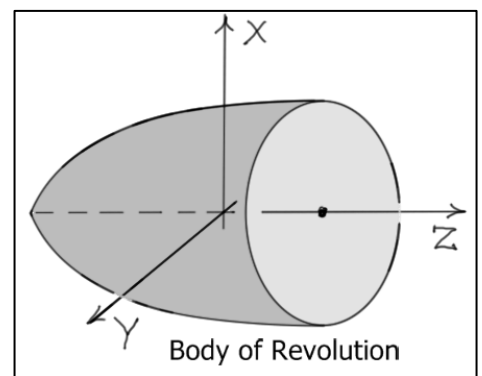
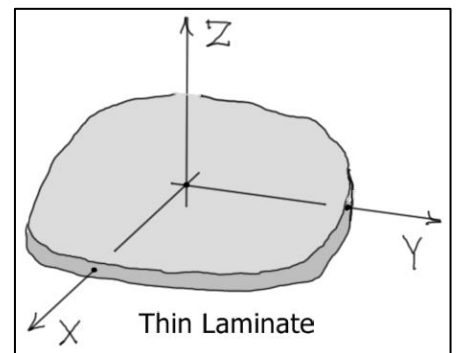
- **Products of inertia** of the rigid body are defined as

$$\boxed{I_{xy}^G = \int_B (x y) dm} \quad \boxed{I_{xz}^G = \int_B (x z) dm}, \quad \boxed{I_{yz}^G = \int_B (y z) dm}$$

- Products of inertia of a body are **measures of body symmetry**.
- If a particular plane is a **plane of symmetry**, then the products of inertia associated with any axis **perpendicular** to that plane are **zero**.
- For example, consider a **thin laminate**. The mid-plane of the laminate lies in the XY -plane so that half its thickness is above the plane and half is below. Hence, the XY -plane is a plane of symmetry and

$$\boxed{I_{xz} = I_{yz} = 0}$$

- **Bodies of revolution** have **two planes of symmetry**. For the configuration shown, the XZ and YZ planes are planes of symmetry. Hence, **all products** of inertia **are zero** about the X , Y , and Z axes.



- Products of inertia are found either by *measurement* or *calculation*. Calculations are based on direct integration or on the “*body build-up*” technique. In the body build-up technique, products of inertia of *simple shapes* are *added* (or subtracted) to estimate the products of inertia of a *composite shape*.
- The products of inertia of simple shapes (about their individual mass centers) are found in *standard inertia tables*. These values are *transferred* to axes through the composite mass center using the *Parallel Axes Theorem for Products of Inertia*.

Parallel Axes Theorem for Products of Inertia

- The product of inertia (I_{ij}^A) of a body about a pair of axes (i, j) passing through any point (A) is equal to the product of inertia (I_{ij}^G) of the body about a set of parallel axes through the mass center G plus the mass (m) times the product of the coordinates ($c_i c_j$) of G relative to A measured along those axes.

$$\boxed{I_{ij}^A = I_{ij}^G + m c_i c_j} \quad (i, j = x, y, \text{ or } z)$$

- *Products of inertia can be positive, negative, or zero.*

The Inertia Matrix

- The inertias of a body about a set of axes (passing through some point) are collected into a single inertia matrix. For example, the inertia matrix of a body about a set of axes through its mass center G is defined as

$$\boxed{[I_G] = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix}}$$

- There is a *different inertia matrix for each set of axes* passing through a given point.
- There is one set of directions for each point that renders the inertia matrix *diagonal*. These directions are called *principal directions* (or *principal axes*) of the body for that point. In general, the *principal axes* are different for each point in a body.
- All inertia matrices are *symmetric*.