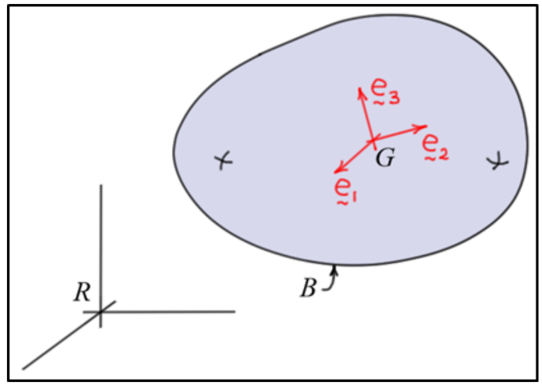
**ME 6590 Multibody Dynamics**
**Examples for Kane's Equations**

## Examples

1. **Unconstrained Motion of a Rigid Body:** Find the EOM of the rigid body using Kane's equations. Use ***Euler parameters*** to describe the orientation of the body and define the generalized speeds as follows




$$\left[u_1,u_2,u_3,u_4,u_5,u_6\right]=\left[\dot{x}_1'^G,\dot{x}_2'^G,\dot{x}_3'^G,\omega_1',\omega_2',\omega_3'\right]$$

where the $\dot{x}_i'^G$ $(i=1,2,3)$ represent the ***body-fixed***, mass-center velocity components, and the $\omega_i'$ $(i=1,2,3)$ represent the ***body-fixed*** angular velocity components.

Solution:

Letting the $\underset{\sim}{e}_i$ $(i=1,2,3)$ represent the ***principal directions*** for the mass-center $G$, and using ***body-fixed*** components of $\underset{\sim}{\omega}_B$ and $\underset{\sim}{v}_G$, write

$$\underset{\sim}{v}_G=\sum_{i=1}^{3}\dot{x}_i'^G\underset{\sim}{e}_i\triangleq\sum_{i=1}^{3}v_i'\underset{\sim}{e}_i \qquad \underset{\sim}{\omega}_B=\sum_{i=1}^{3}\omega_i'\underset{\sim}{e}_i$$

$$\partial\underset{\sim}{v}_G/\partial\dot{x}_i'^G=\underset{\sim}{e}_i \qquad \partial\underset{\sim}{v}_G/\partial\omega_i'=0 \qquad \partial\underset{\sim}{\omega}_B/\partial\omega_i'=\underset{\sim}{e}_i \qquad \partial\underset{\sim}{\omega}_B/\partial v_i'=0$$

$$\underset{\sim}{a}_G=\sum_{i=1}^{3}\ddot{x}_i'^G\underset{\sim}{e}_i=\sum_{i=1}^{3}a_i'\underset{\sim}{e}_i=\left(\dot{v}_1'+\omega_2'v_3'-\omega_3'v_2'\right)\underset{\sim}{e}_1+\left(\dot{v}_2'+\omega_3'v_1'-\omega_1'v_3'\right)\underset{\sim}{e}_2+\left(\dot{v}_3'+\omega_1'v_2'-\omega_2'v_1'\right)\underset{\sim}{e}_3$$

Terms in the Equations of Motion:

$$m\underset{\sim}{a}_G\cdot\left(\partial\underset{\sim}{v}_G/\partial\dot{x}_i'^G\right)=ma_i' \qquad \left(\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B\right)\cdot\partial\underset{\sim}{\omega}_B/\partial\omega_i'=\left(\sum_{j=1}^{3}I_j\dot{\omega}_j'\underset{\sim}{e}_j\right)\cdot\underset{\sim}{e}_i=I_i\dot{\omega}_i'$$

$$\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\partial\underset{\sim}{\omega}_B/\partial\omega_i'=\left((I_3-I_2)\omega_2'\omega_3'\underset{\sim}{e}_1+(I_1-I_3)\omega_1'\omega_3'\underset{\sim}{e}_2+(I_2-I_1)\omega_1'\omega_2'\underset{\sim}{e}_3\right)\cdot\underset{\sim}{e}_i$$

$$=\begin{cases}(I_3-I_2)\omega_2'\omega_3' & (i=1)\\(I_1-I_3)\omega_1'\omega_3' & (i=2)\\(I_2-I_1)\omega_1'\omega_2' & (i=3)\end{cases}$$

Equations of Motion:

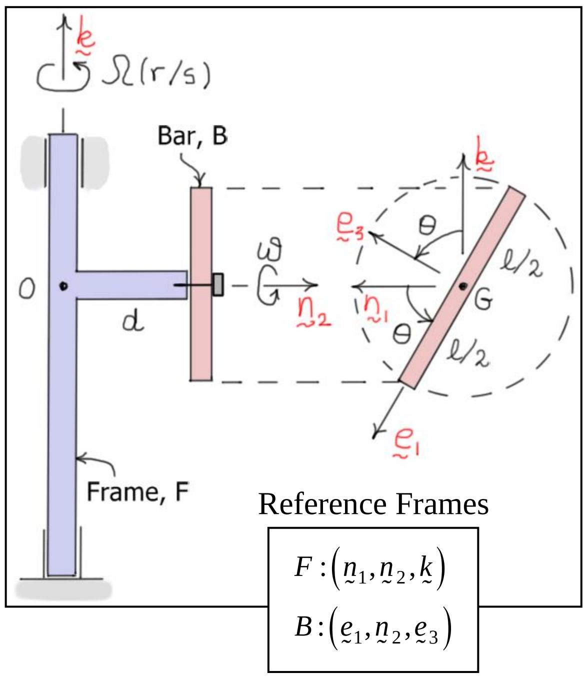
$$\boxed{\begin{aligned} m(\dot{v}_1' + \omega_2' v_3' - \omega_3' v_2') &= F_{v_1'} \\ m(\dot{v}_2' + \omega_3' v_1' - \omega_1' v_3') &= F_{v_2'} \\ m(\dot{v}_3' + \omega_1' v_2' - \omega_2' v_1') &= F_{v_3'} \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} I_1\dot{\omega}_1' + (I_3 - I_2)\omega_2'\omega_3' &= F_{\omega_1} \\ I_2\dot{\omega}_2' + (I_1 - I_3)\omega_1'\omega_3' &= F_{\omega_2} \\ I_3\dot{\omega}_3' + (I_2 - I_1)\omega_1'\omega_2' &= F_{\omega_3} \end{aligned}} \qquad (2)$$

Eqs. (2) are now ***supplemented*** with the ***kinematic differential equations***.

$$\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}\begin{Bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{x}_1'^G \\ \dot{x}_2'^G \\ \dot{x}_3'^G \end{Bmatrix} = \begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix} \qquad (3)$$

Together, Eqs. (2)-(3) represent a ***set*** of ***thirteen first-order***, ***ordinary differential equations*** for the ***four*** Euler parameters, ***three*** mass-center position coordinates, and the ***six*** generalized speeds defined by Eq. (1).

**2. Example System II (from ME 5550):** Find the EOM of the bar using Kane's equations, given that the frame $F$ is light (massless), the bar $B$ has mass $m$ and length $\ell$, the motor torque $M_\theta(t)$ is applied between the frame and the bar, and that the motor torque $M_\phi(t)$ is applied between the ground and the frame. Use $(u_1, u_2) = (v_G, \omega_2')$, where $v_G = -\underset{\sim}{n}_1 \cdot \underset{\sim}{v}_G$ and $\omega_2' = \dot{\theta} = \underset{\sim}{\omega}_B \cdot \underset{\sim}{n}_2$ as the two ***independent generalized speeds***.




Solution: Using Kane's Equations

$$\boxed{\begin{aligned} \left(m_B\, \underset{\sim}{a}_{G_B} \cdot \frac{\partial \underset{\sim}{v}_{G_B}}{\partial v_G}\right) + \left[\left(\underset{\approx}{I}_{G_B} \cdot \underset{\sim}{\alpha}_B\right) + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_{G_B}\right)\right] \cdot \frac{\partial \underset{\sim}{\omega}_B}{\partial v_G} &= F_{v_G} \\ \left(m_B\, \underset{\sim}{a}_{G_B} \cdot \frac{\partial \underset{\sim}{v}_{G_B}}{\partial \omega_2'}\right) + \left[\left(\underset{\approx}{I}_{G_B} \cdot \underset{\sim}{\alpha}_B\right) + \left(\underset{\sim}{\omega}_B \times \underset{\sim}{H}_{G_B}\right)\right] \cdot \frac{\partial \underset{\sim}{\omega}_B}{\partial \omega_2'} &= F_{\omega_2'} \end{aligned}} \qquad (4)$$

where

$$\underset{\sim}{\omega}_F = (v_G/d)\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}_F/\partial v_G = \underset{\sim}{k}/d \qquad \partial\underset{\sim}{\omega}_F/\partial\omega_2' = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_B = \omega_2'\underset{\sim}{n}_2 + (v_G/d)\underset{\sim}{k} \qquad \partial\underset{\sim}{\omega}_B/\partial v_G = \underset{\sim}{k}/d \qquad \partial\underset{\sim}{\omega}_B/\partial\omega_2' = \underset{\sim}{n}_2 = \underset{\sim}{e}_2$$

$$\underset{\sim}{v}_G = -v_G\underset{\sim}{n}_1 \qquad \partial\underset{\sim}{v}_G/\partial v_G = -\underset{\sim}{n}_1 \qquad \partial\underset{\sim}{v}_G/\partial\omega_2' = \underset{\sim}{0}$$

$$\underset{\sim}{a}_G = -\dot{v}_G\underset{\sim}{n}_1 - (v_G^2/d)\underset{\sim}{n}_2 \quad \text{(normal and tangential components)}$$

$$\underset{\sim}{\alpha}_F = (\dot{v}_G/d)\underset{\sim}{k}$$

$$\underset{\sim}{\alpha}_B = -\tfrac{1}{d}(\dot{v}_G S_\theta + v_G\omega_2' C_\theta)\underset{\sim}{e}_1 + \dot{\omega}_2'\underset{\sim}{e}_2 + \tfrac{1}{d}(\dot{v}_G C_\theta - v_G\omega_2' S_\theta)\underset{\sim}{e}_3$$

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$$m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial v_G) = m\dot{v}_G \qquad m\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial\omega_2') = 0$$

$$\underset{\sim}{H}_G = \tfrac{1}{12}m\ell^2(\omega_2'\underset{\sim}{e}_2 + (v_G C_\theta/d)\underset{\sim}{e}_3)$$

$$\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B = \tfrac{1}{12}m\ell^2\left(\dot{\omega}_2'\underset{\sim}{e}_2 + \tfrac{1}{d}(\dot{v}_G C_\theta - v_G\omega_2' S_\theta)\underset{\sim}{e}_3\right)$$

$$\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G = \tfrac{1}{12}m\ell^2\left((v_G^2 S_\theta C_\theta/d^2)\underset{\sim}{e}_2 - (v_G\omega_2' S_\theta/d)\underset{\sim}{e}_3\right)$$

$$\left(\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial v_G\right) = \tfrac{m\ell^2}{12d^2}(\dot{v}_G C_\theta - v_G\,\omega_2'\,S_\theta)C_\theta$$

$$\left(\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\omega_2'\right) = \tfrac{1}{12}m\ell^2\dot{\omega}_2'$$

$$\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial v_G\right) = -\tfrac{m\ell^2}{12d^2}v_G\,\omega_2'\,S_\theta C_\theta$$

$$\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\omega_2'\right) = \tfrac{m\ell^2}{12d^2}v_G^2\,S_\theta\,C_\theta$$

$$F_{v_G} = \left(M_\theta\underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial v_G\right) + \left(-M_\theta\underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial v_G\right) + \left(M_\phi\underset{\sim}{k}\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial v_G\right) \;=\; M_\phi/d$$

$$F_{\omega_2'} = \left(M_\theta\underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\omega_2'\right) + \left(-M_\theta\underset{\sim}{n}_2\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\omega_2'\right) + \left(M_\phi\underset{\sim}{k}\right)\cdot\left(\partial\underset{\sim}{\omega}_F/\partial\omega_2'\right) \;=\; M_\theta$$

Substituting into Kane's equations gives the two differential EOM

$$\boxed{\begin{aligned} \left(md + \tfrac{mL^2}{12d}C_\theta^2\right)\dot{v}_G - \left(\tfrac{mL^2}{6d}S_\theta C_\theta\right)v_G\omega_2' &= M_\phi \\ \left(\tfrac{mL^2}{12}\right)\dot{\omega}_2' + \left(\tfrac{mL^2}{12d^2}S_\theta C_\theta\right)v_G^2 &= M_\theta \end{aligned}} \tag{5}$$

Eqs. (5) are now ***supplemented*** with the ***kinematic differential equations***.

$$\boxed{\dot{\theta} = \omega_2'} \quad \text{and} \quad \boxed{\dot{\phi} = v_G/d}$$