ME 6590 Multibody Dynamics
Principal Moments of Inertia and Principal Directions

Properties of Real Symmetric Matrices

(Reference: Richard Bronson, *Matrix Methods: An Introduction*, Academic Press, 1970.)

All *real, symmetric matrices* have the following *properties*:

- o The *eigenvalues* of a real, symmetric matrix are *real*.

- o The *eigenvectors* of a real, symmetric matrix can always be chosen to be *real*.

- o A real, symmetric matrix is *diagonalizable*.

- o Every real, symmetric matrix possesses a *complete set* of *orthonormal eigenvectors*.

- o For every real symmetric $n \times n$ matrix $[A]$, there exists an $n \times n$ *real orthogonal matrix* $[M]$ such that $[M]^T[A][M] = [D]$, where $[D]$ is a diagonal matrix.

- o $[M]$ is called the *modal matrix*. Its columns are formed by the *eigenvectors* of $[A]$.

- o $[D]$ is a diagonal matrix whose entries are the *eigenvalues* of $[A]$.

- o *Eigenvalues* appear in the *same columns* of $[D]$ as their *associated eigenvectors* appear in $[M]$.

Principal Moments of Inertia and Principal Directions

- o The *inertia dyadic* of the body for a set of axes passing through its mass-center and parallel to the unit vector set $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ is defined as

$$\underset{\approx}{I}'_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I'_{ij}\, \underset{\sim}{e}_i\, \underset{\sim}{e}_j$$
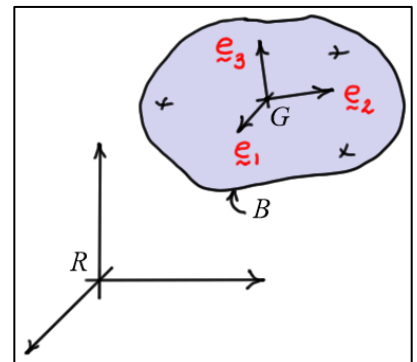
- o The *components* of the inertia dyadic form a *real, symmetric* 3×3 *matrix* $[I'_G]$. Consequently, this matrix has all the properties listed above.

- o The eigenvectors of $[I'_G]$ define the *principal directions* of the body for the mass-center G. Recall that the *principal directions* are those directions for which *all products of inertia* are *zero*.

o The *eigenvalues* of $[I'_G]$ are the *principal moments of inertia* of the body for G, that is, they are the *moments of inertia* about the *principal directions*.

o Note that the *principal directions* and inertia *vary* from point to point in the body, but *there exists only one set of principal directions for any given point*.

Calculation of Principal Moments of Inertia and Principal Directions

o The *principal moments of inertia* of a body for a given point, say the mass-center G, can be calculated (as you would to find the eigenvalues of any 3×3 matrix) by setting

$$\det \begin{bmatrix} (I'_{xx} - \lambda) & -I'_{xy} & -I'_{xz} \\ -I'_{xy} & (I'_{yy} - \lambda) & -I'_{yz} \\ -I'_{xz} & -I'_{yz} & (I'_{zz} - \lambda) \end{bmatrix} = 0. \tag{1}$$

By *expanding* the *determinant*, the resulting characteristic equation can be written as

$$\lambda^3 + (-I'_{xx} - I'_{yy} - I'_{zz})\lambda^2 + (I'_{xx}I'_{yy} + I'_{xx}I'_{zz} + I'_{yy}I'_{zz} - I'^2_{xy} - I'^2_{xz} - I'^2_{yz})\lambda$$
$$+ (-I'_{xx}I'_{yy}I'_{zz} + I'_{xx}I'^2_{yz} + I'_{yy}I'^2_{xz} + I'_{zz}I'^2_{xy} + 2I'_{xy}I'_{xz}I'_{yz}) = 0 \tag{2}$$

The *three roots* to this equation are the *three principal moments of inertia*.

o If I_i $(i = 1, 2, 3)$ represent the three principal moments of inertia, the principal directions can be found (as you would find the eigenvectors of any 3×3 matrix) by writing

$$\begin{bmatrix} (I'_{xx} - I_i) & -I'_{xy} & -I'_{xz} \\ -I'_{xy} & (I'_{yy} - I_i) & -I'_{yz} \\ -I'_{xz} & -I'_{yz} & (I'_{zz} - I_i) \end{bmatrix} \begin{Bmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{Bmatrix} = \{0\} \tag{3}$$

o Since the coefficient matrix is *singular*, these equations *do not have a single solution*. Eigenvectors can be found to within a constant multiplier only.

o If the eigenvectors are defined to be *unit vectors*, then $\boxed{a^2_{i1} + a^2_{i2} + a^2_{i3} = 1}$. In this case, the eigenvectors are unique. Recall that the components of a unit vector are the direction cosines for that direction.

o To solve Eq. (3), simply choose a value for one of the a_{ij} $(j = 1, 2, 3)$, and then solve for the other two. Then, *normalize* the resulting vector.