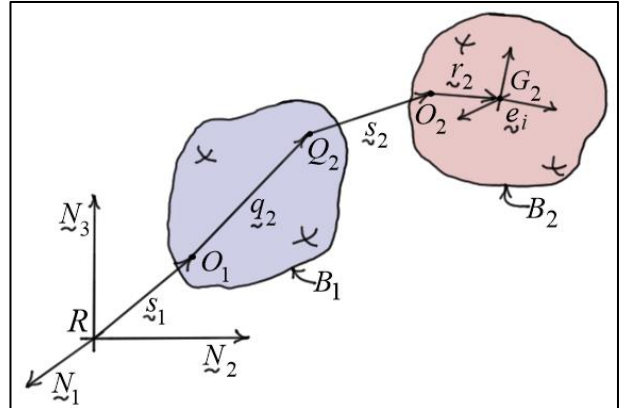


ME 6590 Multibody Dynamics

Velocities and Partial Velocities Using Relative Coordinates

Velocities

The motions of the mass centers of the bodies can also be determined by working *through the other bodies* of the system. For example, the fixed-frame components of the *position vector* of the mass center G_2 can be written as



$$\begin{aligned} \{p_{G_2}\} &= \{s_1\} + \{q_2\} + \{s_2\} + \{r_2\} \\ &= \{s_1\} + [R_{B_1}]^T \{q'_2\} + [R_{B_1}]^T \{s'_2\} + [R_{B_2}]^T \{r'_2\} \end{aligned}$$

where it has been assumed that $\{s_1\}$ is given in the fixed frame, $\{q'_2\}$ and $\{s'_2\}$ are given in the B_1 frame, and $\{r'_2\}$ is given in the B_2 frame.

The *velocity* of G_2 can be found by *differentiating* the above equation.

$$\{v_{G_2}\} = \{\dot{s}_1\} + [\dot{R}_{B_1}]^T \{q'_2\} + [\dot{R}_{B_1}]^T \{s'_2\} + [R_{B_1}]^T \{\dot{s}'_2\} + [\dot{R}_{B_2}]^T \{r'_2\}$$

To expand this equation further, first decide whether to use *fixed-frame* or *body-frame components* for the *angular velocity vectors*. The two results are as follows.

$$\boxed{\{v_{G_2}\} = \{\dot{s}_1\} + [\tilde{\omega}_{B_1}] [R_{B_1}]^T (\{q'_2\} + \{s'_2\}) + [R_{B_1}]^T \{\dot{s}'_2\} + [\tilde{\omega}_{B_2}] [R_{B_2}]^T \{r'_2\}} \quad (1)$$

$$\boxed{\{v_{G_2}\} = \{\dot{s}_1\} + [R_{B_1}]^T [\tilde{\omega}'_{B_1}] (\{q'_2\} + \{s'_2\}) + [R_{B_1}]^T \{\dot{s}'_2\} + [R_{B_2}]^T [\tilde{\omega}'_{B_2}] \{r'_2\}} \quad (2)$$

The velocities of *other points* of B_2 are found by simply replacing the coordinates $\{r'_2\}$ with the coordinates of that point.

To find the *partial velocities* of G_2 , first *rearrange* Eqs. (1) and (2). As a first step in this process, note that ${}^R\omega_B \times \underline{r} = -\underline{r} \times {}^R\omega_B$. This vector identity can be written in matrix form as $[\tilde{\omega}_B]\{r\} = -[\tilde{r}]\{\omega_B\}$. Using this result in Eqs. (1) and (2), write

$$\begin{aligned}
\{v_{G_2}\} &= \{\dot{s}_1\} + [\tilde{\omega}_{B_1}][R_{B_1}]^T(\{q'_2\} + \{s'_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [\tilde{\omega}_{B_2}][R_{B_2}]^T\{r'_2\} \\
&= \{\dot{s}_1\} + [\tilde{\omega}_{B_1}](\{q_2\} + \{s_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [\tilde{\omega}_{B_2}]\{r_2\} \\
&= \{\dot{s}_1\} - ([\tilde{q}_2] + [\tilde{s}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2]\{\omega_{B_2}\} \\
&= \{\dot{s}_1\} - ([\tilde{q}_2] + [\tilde{s}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2](\{\omega_{B_1}\} + [R_{B_1}]^T\{\hat{\omega}_{B_2}\}) \\
&= \{\dot{s}_1\} - ([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2])\{\omega_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [\tilde{r}_2][R_{B_1}]^T\{\hat{\omega}_{B_2}\}
\end{aligned} \tag{3}$$

Recall here that the entries of $\{\omega_{B_1}\}$ are expressed in the fixed frame, and the entries of $\{\hat{\omega}_{B_2}\}$ are expressed in the B_1 frame. In both cases, these are the frames in which the angular velocities are measured.

$$\begin{aligned}
\{v_{G_2}\} &= \{\dot{s}_1\} + [R_{B_1}]^T[\tilde{\omega}'_{B_1}](\{q'_2\} + \{s'_2\}) + [R_{B_1}]^T\{\dot{s}'_2\} + [R_{B_2}]^T[\tilde{\omega}'_{B_2}]\{r'_2\} \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2] + [\tilde{s}'_2])\{\omega'_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} - [R_{B_2}]^T[\tilde{r}'_2]\{\omega'_{B_2}\} \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2] + [\tilde{s}'_2])\{\omega'_{B_1}\} + [R_{B_1}]^T\{\dot{s}'_2\} \\
&\quad - [R_{B_2}]^T[\tilde{r}'_2](\{^{B_2}R_{B_1}\}^T\{\omega'_{B_1}\} + \{\hat{\omega}'_{B_2}\}) \\
&= \{\dot{s}_1\} - [R_{B_1}]^T([\tilde{q}'_2] + [\tilde{s}'_2])\{\omega'_{B_1}\} - [R_{B_2}]^T[\tilde{r}'_2][^{B_2}R_{B_1}]^T\{\omega'_{B_1}\} \\
&\quad + [R_{B_1}]^T\{\dot{s}'_2\} - [R_{B_2}]^T[\tilde{r}'_2]\{\hat{\omega}'_{B_2}\}
\end{aligned} \tag{4}$$

Partial Velocities

Eqs. (3) and (4) can be used to find the *partial velocities* of the mass centers of the bodies.

Case 1:

If the *generalized speeds* are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\{\omega_{B_1}\}$, and $\{\hat{\omega}_{B_2}\}$, then

$$\left\{v_{G_2}\right\} = \left[v_{G_2, \dot{s}_1}\right] \left\{\dot{s}_1\right\} + \left[v_{G_2, \omega_{B_1}}\right] \left\{\omega_{B_1}\right\} + \left[v_{G_2, \dot{s}'_2}\right] \left\{\dot{s}'_2\right\} + \left[v_{G_2, \hat{\omega}_{B_2}}\right] \left\{\hat{\omega}_{B_2}\right\}$$

where

$\left[v_{G_2, \dot{s}_1}\right]$ is a 3×3 identity matrix

$$\left[v_{G_2, \omega_{B_1}}\right] = -\left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)$$

$$\left[v_{G_2, \dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2, \hat{\omega}_{B_2}}\right] = -\left[\tilde{r}_2\right] \left[R_{B_1}\right]^T$$

Case 2:

If the *generalized speeds* are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\{\omega'_{B_1}\}$, and $\{\hat{\omega}'_{B_2}\}$, then

$$\left\{v_{G_2}\right\} = \left[v_{G_2, \dot{s}_1}\right] \left\{\dot{s}_1\right\} + \left[v_{G_2, \omega'_{B_1}}\right] \left\{\omega'_{B_1}\right\} + \left[v_{G_2, \dot{s}'_2}\right] \left\{\dot{s}'_2\right\} + \left[v_{G_2, \hat{\omega}'_{B_2}}\right] \left\{\hat{\omega}'_{B_2}\right\}$$

where

$\left[v_{G_2, \dot{s}_1}\right]$ is a 3×3 identity matrix

$$\left[v_{G_2, \omega'_{B_1}}\right] = -\left[R_{B_1}\right]^T \left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right) - \left[R_{B_2}\right]^T \left[\tilde{r}'_2\right] \left[{}^{B_2}R_{B_1}\right]^T$$

$$\left[v_{G_2, \dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2, \hat{\omega}'_{B_2}}\right] = -\left[R_{B_2}\right]^T \left[\tilde{r}'_2\right]$$

Case 3:

If, instead, the generalized speeds are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}_2\}$, and a set of *orientation angle derivatives*, then we can write $\{\omega_{B_1}\}$ and $\{\hat{\omega}_{B_2}\}$ (or $\{\omega'_{B_1}\}$ and $\{\hat{\omega}'_{B_2}\}$) in terms of the orientation angles and then identify the partial velocities. Returning to Eq. (3), for example, write

$$\begin{aligned}
 \{v_{G_2}\} &= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2] \right) \{\omega_{B_1}\} + [R_{B_1}]^T \{\dot{s}_2\} - [\tilde{r}_2] [R_{B_1}]^T \{\hat{\omega}_{B_2}\} \\
 &= \{\dot{s}_1\} - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2] \right) [\omega_{B_1, \dot{\theta}_{B_1}}] \{\dot{\theta}_{B_1}\} + [R_{B_1}]^T \{\dot{s}_2\} \\
 &\quad - [\tilde{r}_2] [R_{B_1}]^T [\hat{\omega}_{B_2, \dot{\theta}_{B_2}}] \{\dot{\theta}_{B_2}\} \\
 &= [v_{G_2, \dot{s}_1}] \{\dot{s}_1\} + [v_{G_2, \dot{\theta}_{B_1}}] \{\dot{\theta}_{B_1}\} + [v_{G_2, \dot{s}'_2}] \{\dot{s}'_2\} + [v_{G_2, \dot{\theta}_{B_2}}] \{\dot{\theta}_{B_2}\}
 \end{aligned} \tag{5}$$

where

$$\begin{aligned}
 [v_{G_2, \dot{s}_1}] &\text{ is a } 3 \times 3 \text{ identity matrix} \\
 [v_{G_2, \dot{\theta}_{B_1}}] &= - \left([\tilde{q}_2] + [\tilde{s}_2] + [\tilde{r}_2] \right) [\omega_{B_1, \dot{\theta}_{B_1}}] \\
 [v_{G_2, \dot{s}'_2}] &= [R_{B_1}]^T \\
 [v_{G_2, \dot{\theta}_{B_2}}] &= - [\tilde{r}_2] [R_{B_1}]^T [\hat{\omega}_{B_2, \dot{\theta}_{B_2}}]
 \end{aligned}$$

The partial angular velocity matrices are defined in previous notes for a 1-2-3 rotation sequence.