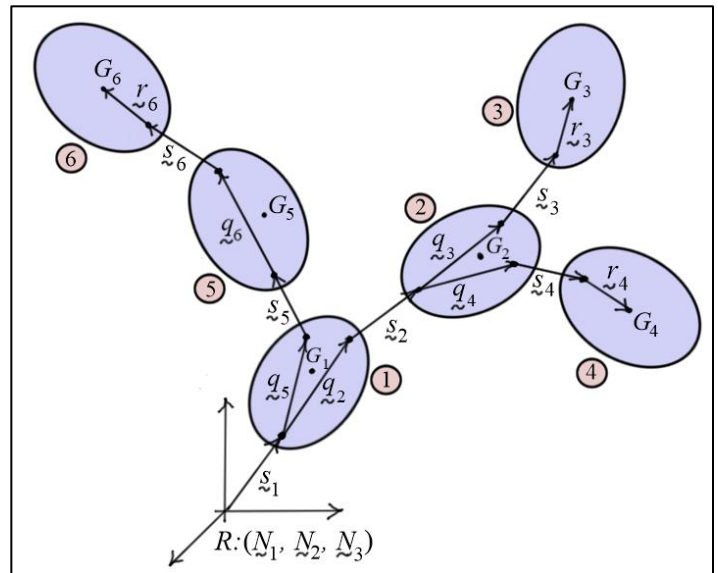


## ME 6590 Multibody Dynamics

### Equations of Motion for an Unconstrained Multibody – Example

The results below all apply to the *six-body* system shown. The *orientations* of the bodies relative to the *inertial system* are described using *Euler parameters*, and the *translation* of the bodies *relative* to their *lower-numbered body* are defined by components of the vectors  $\underline{s}_K$  in the reference frame  $\mathcal{L}(K)$ . The *angular velocities* of the bodies are all measured *relative* to the *inertial frame* and resolved in the body frames.



In total, there are 42 first-order, kinematical differential equations and 36 first-order, dynamical differential equations.

#### Body Connection Array

Using the numbering of the bodies shown on the diagram, the body connection array (or lower-numbered body array) can be written as

$$\mathcal{L}(K) = (0, 1, 2, 2, 1, 5)$$

Recall that *zero* represents the inertial frame. The array  $[u_K]$  is

$$(u_K) = (0, 1, 2, 2, 1, 2)$$

Recall that for each body,  $u_K$  is that integer such that  $\mathcal{L}^{u_K}(K) = 1$ . In other words, it is the number of steps down from body  $K$  to reach the system's reference body (body 1).

#### System State Vectors

$$\{\mathcal{E}\}_{24 \times 1} = [\mathcal{E}_{11}, \mathcal{E}_{12}, \mathcal{E}_{13}, \mathcal{E}_{14}, \dots, \mathcal{E}_{61}, \mathcal{E}_{62}, \mathcal{E}_{63}, \mathcal{E}_{64}]^T \quad \{s'\}_{18 \times 1} = [s'_{11}, s'_{12}, s'_{13}, \dots, s'_{61}, s'_{62}, s'_{63}]^T$$

$$\{\omega'\}_{18 \times 1} = [\omega'_{11}, \omega'_{12}, \omega'_{13}, \dots, \omega'_{61}, \omega'_{62}, \omega'_{63}]^T$$

$$\{x\}_{42 \times 1} = \begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} \{\mathcal{E}\} \\ \{s'\} \end{Bmatrix} \quad \{y\}_{36 \times 1} = \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} = \begin{Bmatrix} \{\omega'\} \\ \{\dot{s}'\} \end{Bmatrix}$$

### System Partial Angular Velocity Matrices

$$\begin{aligned} [\omega'_{1,y_1}]_{3 \times 18} &= \begin{bmatrix} [I], [0], \dots, [0] \\ 1 \quad 2 \quad 6 \end{bmatrix} \quad \text{and} \quad [\omega'_{1,y_2}]_{3 \times 18} = [0] \\ [\omega'_{2,y_1}]_{3 \times 18} &= \begin{bmatrix} [0], [I], [0], \dots, [0] \\ 1 \quad 2 \quad 3 \quad 6 \end{bmatrix} \quad \text{and} \quad [\omega'_{2,y_2}]_{3 \times 18} = [0] \\ [\omega'_{3,y_1}]_{3 \times 18} &= \begin{bmatrix} [0], [0], [I], [0], \dots, [0] \\ 1 \quad 2 \quad 3 \quad 4 \quad 6 \end{bmatrix} \quad \text{and} \quad [\omega'_{3,y_2}]_{3 \times 18} = [0] \\ &\vdots \end{aligned}$$

where  $[I]$  represents a  $3 \times 3$  identity matrix, and  $[0]$  represents a  $3 \times 3$  zero matrix.

### Position Vectors of the Mass Centers of the Bodies

$$\begin{aligned} \{p_1\} &= \{s_1\} + [R_1]^T \{r'_1\} \\ \{p_2\} &= \{s_1\} + [R_1]^T \{q'_2 + s'_2\} + [R_2]^T \{r'_2\} \\ \{p_3\} &= \{s_1\} + [R_1]^T \{q'_2 + s'_2\} + [R_2]^T \{q'_3 + s'_3\} + [R_3]^T \{r'_3\} \\ \{p_4\} &= \{s_1\} + [R_1]^T \{q'_2 + s'_2\} + [R_2]^T \{q'_4 + s'_4\} + [R_4]^T \{r'_4\} \\ \{p_5\} &= \{s_1\} + [R_1]^T \{q'_5 + s'_5\} + [R_5]^T \{r'_5\} \\ \{p_6\} &= \{s_1\} + [R_1]^T \{q'_5 + s'_5\} + [R_5]^T \{q'_6 + s'_6\} + [R_6]^T \{r'_6\} \end{aligned}$$

### Velocity Vectors of the Mass Centers of the Bodies

$$\begin{aligned} \{v_1\} &= \{\dot{s}_1\} - [R_1]^T [\tilde{r}'_1] [\omega'_{1,y}] \{y\} \\ \{v_2\} &= \{\dot{s}_1\} + [R_1]^T \{\dot{s}'_2\} - [R_1]^T [\tilde{q}'_2 + \tilde{s}'_2] [\omega'_{1,y}] \{y\} - [R_2]^T [\tilde{r}'_2] [\omega'_{2,y}] \{y\} \\ \{v_3\} &= \{\dot{s}_1\} + [R_1]^T \{\dot{s}'_2\} + [R_2]^T \{\dot{s}'_3\} - \\ &\quad [R_1]^T [\tilde{q}'_2 + \tilde{s}'_2] [\omega'_{1,y}] \{y\} - [R_2]^T [\tilde{q}'_3 + \tilde{s}'_3] [\omega'_{2,y}] \{y\} - [R_3]^T [\tilde{r}'_3] [\omega'_{3,y}] \{y\} \end{aligned}$$

$$\begin{aligned} \{v_4\} &= \{\dot{s}_1\} + [R_1]^T \{\dot{s}'_2\} + [R_2]^T \{\dot{s}'_4\} - \\ &\quad [R_1]^T [\tilde{q}'_2 + \tilde{s}'_2][\omega'_{1,y}\{y\}] - [R_2]^T [\tilde{q}'_4 + \tilde{s}'_4][\omega'_{2,y}\{y\}] - [R_4]^T [\tilde{r}'_4][\omega'_{4,y}\{y\}] \\ \{v_5\} &= \{\dot{s}_1\} + [R_1]^T \{\dot{s}'_5\} - [R_1]^T [\tilde{q}'_5 + \tilde{s}'_5][\omega'_{1,y}\{y\}] - [R_5]^T [\tilde{r}'_5][\omega'_{5,y}\{y\}] \\ \{v_6\} &= \{\dot{s}_1\} + [R_1]^T \{\dot{s}'_5\} + [R_5]^T \{\dot{s}'_6\} - \\ &\quad [R_1]^T [\tilde{q}'_5 + \tilde{s}'_5][\omega'_{1,y}\{y\}] - [R_5]^T [\tilde{q}'_6 + \tilde{s}'_6][\omega'_{5,y}\{y\}] - [R_6]^T [\tilde{r}'_6][\omega'_{6,y}\{y\}] \end{aligned}$$

### System Partial Velocity Matrices

$$\begin{aligned} [v_{1,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{r}'_1], [0], [0], [0], [0], [0]] \\ [v_{2,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{q}'_2 + \tilde{s}'_2], -[R_2]^T [\tilde{r}'_2], [0], [0], [0], [0]] \\ [v_{3,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{q}'_2 + \tilde{s}'_2], -[R_2]^T [\tilde{q}'_3 + \tilde{s}'_3], -[R_3]^T [\tilde{r}'_3], [0], [0], [0]] \\ [v_{4,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{q}'_2 + \tilde{s}'_2], -[R_2]^T [\tilde{q}'_4 + \tilde{s}'_4], [0], -[R_4]^T [\tilde{r}'_4], [0], [0]] \\ [v_{5,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{q}'_5 + \tilde{s}'_5], [0], [0], [0], -[R_5]^T [\tilde{r}'_5], [0]] \\ [v_{6,y_1}]_{3 \times 18} &= [-[R_1]^T [\tilde{q}'_5 + \tilde{s}'_5], [0], [0], [0], -[R_5]^T [\tilde{q}'_6 + \tilde{s}'_6], -[R_6]^T [\tilde{r}'_6]] \end{aligned}$$

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$$\begin{aligned} [v_{1,y_2}]_{3 \times 18} &= [I, [0], [0], [0], [0], [0]] \\ [v_{2,y_2}]_{3 \times 18} &= [I, [R_1]^T, [0], [0], [0], [0]] \\ [v_{3,y_2}]_{3 \times 18} &= [I, [R_1]^T, [R_2]^T, [0], [0], [0]] \\ [v_{4,y_2}]_{3 \times 18} &= [I, [R_1]^T, [0], [R_2]^T, [0], [0]] \\ [v_{5,y_2}]_{3 \times 18} &= [I, [0], [0], [0], [R_1]^T, [0]] \\ [v_{6,y_2}]_{3 \times 18} &= [I, [0], [0], [0], [R_1]^T, [R_5]^T] \end{aligned}$$

### Time Derivatives of the Partial Angular Velocity Matrices

$$[\dot{\omega}'_{K,y_1}] = [0] \quad \text{and} \quad [\dot{\omega}'_{K,y_2}] = [0] \quad K = 1, \dots, 6$$

## Time Derivatives of the Partial Velocity Matrices

$$\left[ \dot{v}_{1,y_1} \right]_{3 \times 18} = \left[ -[R_1]^T [\tilde{\omega}'_1] [\tilde{r}'_1], [0], [0], [0], [0], [0] \right]$$

$$\left[ \dot{v}_{2,y_1} \right]_{3 \times 18} = \left[ \left( -[R_1]^T [\dot{s}'_2] - [R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_2 + \tilde{s}'_2] \right), -[R_2]^T [\tilde{\omega}'_2] [\tilde{r}'_2], [0], [0], [0], [0] \right]$$

$$\left[ \dot{v}_{3,y_1} \right]_{3 \times 18} = \left[ \left( -[R_1]^T [\dot{s}'_2] - [R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_2 + \tilde{s}'_2] \right), \right. \\ \left. \left( -[R_2]^T [\dot{s}'_3] - [R_2]^T [\tilde{\omega}'_2] [\tilde{q}'_3 + \tilde{s}'_3] \right), -[R_3]^T [\tilde{\omega}'_3] [\tilde{r}'_3], [0], [0], [0] \right]$$

$$\left[ \dot{v}_{4,y_1} \right]_{3 \times 18} = \left[ \left( -[R_1]^T [\dot{s}'_2] - [R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_2 + \tilde{s}'_2] \right), \right. \\ \left. \left( -[R_2]^T [\dot{s}'_4] - [R_2]^T [\tilde{\omega}'_2] [\tilde{q}'_4 + \tilde{s}'_4] \right), [0], -[R_4]^T [\tilde{\omega}'_4] [\tilde{r}'_4], [0], [0] \right]$$

$$\left[ \dot{v}_{5,y_1} \right]_{3 \times 18} = \left[ \left( -[R_1]^T [\dot{s}'_5] - [R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_5 + \tilde{s}'_5] \right), [0], [0], [0], -[R_5]^T [\tilde{\omega}'_5] [\tilde{r}'_5], [0] \right]$$

$$\left[ \dot{v}_{6,y_1} \right]_{3 \times 18} = \left[ \left( -[R_1]^T [\dot{s}'_5] - [R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_5 + \tilde{s}'_5] \right), [0], [0], [0], \right. \\ \left. \left( -[R_5]^T [\dot{s}'_6] - [R_5]^T [\tilde{\omega}'_5] [\tilde{q}'_6 + \tilde{s}'_6] \right), -[R_6]^T [\tilde{\omega}'_6] [\tilde{r}'_6] \right]$$

$$\left[ \dot{v}_{1,y_2} \right]_{3 \times 18} = [0], [0], [0], [0], [0], [0]$$

$$\left[ \dot{v}_{2,y_2} \right]_{3 \times 18} = [0], [R_1]^T [\tilde{\omega}'_1], [0], [0], [0], [0]$$

$$\left[ \dot{v}_{3,y_2} \right]_{3 \times 18} = [0], [R_1]^T [\tilde{\omega}'_1], [R_2]^T [\tilde{\omega}'_2], [0], [0], [0]$$

$$\left[ \dot{v}_{4,y_2} \right]_{3 \times 18} = [0], [R_1]^T [\tilde{\omega}'_1], [0], [R_2]^T [\tilde{\omega}'_2], [0], [0]$$

$$\left[ \dot{v}_{5,y_2} \right]_{3 \times 18} = [0], [0], [0], [0], [R_1]^T [\tilde{\omega}'_1], [0]$$

$$\left[ \dot{v}_{6,y_2} \right]_{3 \times 18} = [0], [0], [0], [0], [R_1]^T [\tilde{\omega}'_1], [R_5]^T [\tilde{\omega}'_5]$$

## Equations of Motion

Given the partial velocity and partial angular velocity matrices above, the dynamical equations of motion can be written as

$$\boxed{[A]\{\dot{y}\} = \{f\}} \quad ([A] \text{ is the generalized mass matrix})$$

where

$$\boxed{[A] = \sum_{K=1}^6 \left( m_K [v_{K,y}]^T [v_{K,y}] + [\omega'_{K,y}]^T [I'_K] [\omega'_{K,y}] \right)}$$

$$\boxed{\{f\} = \sum_{K=1}^6 [v_{K,y}]^T \left( \{F_K\} - m_K [\dot{v}_{K,y}] \{y\} \right) + \sum_{K=1}^6 [\omega'_{K,y}]^T \left( \{M'_K\} - [\tilde{\omega}'_K] [I'_K] \{\omega'_K\} \right)}$$

The above equations represent 36 dynamical differential equations that must be supplemented with 42 kinematical differential equations.