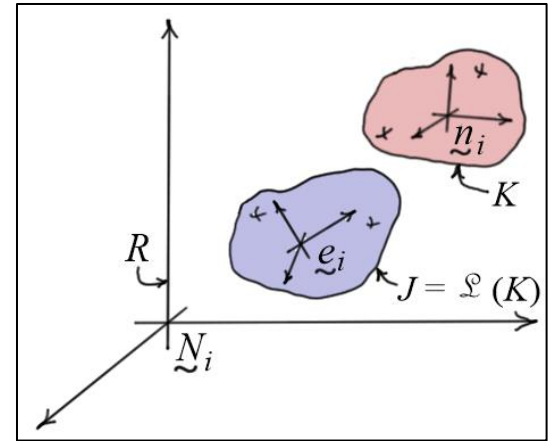


ME 6590 Multibody Dynamics

Time Derivative of Relative Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider two bodies of a multibody system. The unit vector set $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ is fixed in body K , and the set $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is fixed in body J . Body J is the lower-numbered body of K ($J = \mathcal{L}(K)$). Both bodies are moving in a fixed-frame $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$. If \underline{r} is a vector fixed in the body K , then, using the summation rule for angular velocities, the time derivative of \underline{r} can be written as



$$\boxed{\frac{{}^R d\underline{r}}{dt} = \dot{\underline{r}} = {}^R \underline{\omega}_K \times \underline{r} = ({}^R \underline{\omega}_J + {}^J \underline{\omega}_K) \times \underline{r} = ({}^R \underline{\omega}_J \times \underline{r}) + ({}^J \underline{\omega}_K \times \underline{r})} \quad (1)$$

When performing the cross products, the individual vectors and the resulting cross products can be expressed in any reference frame. Two cases are considered below – components of ${}^J \underline{\omega}_K$ in body J (Case 1) and components of ${}^J \underline{\omega}_K$ in body K (Case 2). Components of vectors in body K have been annotated with a “prime”.

The transformation matrices associated with the two bodies ($[R_J]$, $[R_K]$, $[{}^J R_K]$) are defined by the following equations.

$$\boxed{\{\underline{e}\} = [R_J] \{\underline{N}\}} \quad \boxed{\{\underline{n}\} = [R_K] \{\underline{N}\}} \quad \boxed{\{\underline{n}'\} = [{}^J R_K] \{\underline{e}\}}$$

Case 1:

Let $\dot{\underline{r}}$, ${}^R \underline{\omega}_J$, and ${}^R \underline{\omega}_K$ be expressed in $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, ${}^J \underline{\omega}_K$ be expressed in $J: (\underline{e}_1, \underline{e}_2, \underline{e}_3)$, and \underline{r} be expressed in $K: (\underline{n}_1, \underline{n}_2, \underline{n}_3)$. Then,

$$\boxed{\{\underline{r}\} = [R_K]^T \{\underline{r}'\}} \quad \boxed{\{\dot{\underline{r}}\} = [\dot{R}_K]^T \{\underline{r}'\}}$$

$$\boxed{{}^R \underline{\omega}_K \times \underline{r} \rightarrow [\tilde{\omega}_K][R_K]^T \{\underline{r}'\}} \quad \boxed{{}^R \underline{\omega}_J \times \underline{r} \rightarrow [\tilde{\omega}_J][R_K]^T \{\underline{r}'\}}$$

$$\boxed{{}^J\omega_K \times \underline{r} \rightarrow [R_J]^T [{}^J\tilde{\omega}_K] [{}^J R_K]^T \{r'\}}$$

Substituting into Eq. (1) gives

$$\begin{aligned} \boxed{[\dot{R}_K]^T} &= [\tilde{\omega}_K] [R_K]^T = [\tilde{\omega}_J] [R_K]^T + [R_J]^T [{}^J\tilde{\omega}_K] [{}^J R_K]^T \\ \Rightarrow & \boxed{([\tilde{\omega}_K] - [\tilde{\omega}_J]) [R_K]^T = [R_J]^T [{}^J\tilde{\omega}_K] [{}^J R_K]^T} \end{aligned}$$

or

$$\boxed{[R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T) = [{}^J R_K] [{}^J\tilde{\omega}_K]^T [R_J]} \quad (2)$$

Case 2:

Let \dot{r} , ${}^R\omega_J$, and ${}^R\omega_K$ be expressed in $R:(N_1, N_2, N_3)$, and let ${}^J\omega_K$ and \underline{r} be expressed in $K:(n_1, n_2, n_3)$. Then,

$$\begin{aligned} \boxed{\{r\} = [R_K]^T \{r'\}} \quad & \boxed{\{\dot{r}\} = [\dot{R}_K]^T \{r'\}} \\ \boxed{{}^R\omega_K \times \underline{r} \rightarrow [\tilde{\omega}_K] [R_K]^T \{r'\}} \quad & \boxed{{}^R\omega_J \times \underline{r} \rightarrow [\tilde{\omega}_J] [R_K]^T \{r'\}} \\ \boxed{{}^J\omega_K \times \underline{r} \rightarrow [R_K]^T [{}^J\tilde{\omega}'_K] \{r'\}} \end{aligned}$$

Substituting into Eq. (1) gives

$$\begin{aligned} \boxed{[\dot{R}_K]^T} &= [\tilde{\omega}_K] [R_K]^T = [\tilde{\omega}_J] [R_K]^T + [R_K]^T [{}^J\tilde{\omega}'_K] \\ \Rightarrow & \boxed{([\tilde{\omega}_K] - [\tilde{\omega}_J]) [R_K]^T = [R_K]^T [{}^J\tilde{\omega}'_K]} \end{aligned}$$

or

$$\boxed{[R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T) = [{}^J\tilde{\omega}'_K]^T [R_K]} \quad (3)$$

Time Derivative of the Transformation Matrices Between Two Body Frames

The above results can be used to determine *two different forms* of the *time derivative* of the *relative transformation matrix* $[{}^J R_K]$ depending on what reference frame is used to express the relative angular velocity vector ${}^J\omega_K$. First, note the time derivative of $[{}^J R_K]$ can be written as follows.

$$\begin{aligned} \left[{}^J \dot{R}_K \right] &= \frac{{}^R d}{dt} \left[{}^J R_K \right] = \frac{{}^R d}{dt} \left([R_K][R_J]^T \right) = [\dot{R}_K][R_J]^T + [R_K][\dot{R}_J]^T \\ &= [R_K][\tilde{\omega}_K]^T [R_J]^T + [R_K][\tilde{\omega}_J][R_J]^T \end{aligned}$$

As a skew-symmetric matrix, $[\tilde{\omega}_J] = -[\tilde{\omega}_J]^T$, so the above equation can be rewritten as

$$\Rightarrow \boxed{\left[{}^J \dot{R}_K \right] = [R_K] \left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T \right) [R_J]^T} \quad (4)$$

Now, substituting from Eq. (2) into Eq. (4) gives

$$\begin{aligned} \left[{}^J \dot{R}_K \right] &= [R_K] \left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T \right) [R_J]^T \\ &= [{}^J R_K] [{}^J \tilde{\omega}_K]^T [R_J][R_J]^T \\ &\Rightarrow \boxed{\left[{}^J \dot{R}_K \right] = [{}^J R_K] [{}^J \tilde{\omega}_K]^T} \quad (\text{components of } {}^J \omega_K \text{ are resolved in body } J) \end{aligned}$$

And, substituting from Eq. (3) into Eq. (4) gives

$$\begin{aligned} \left[{}^J \dot{R}_K \right] &= [R_K] \left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T \right) [R_J]^T \\ &= [{}^J \tilde{\omega}'_K]^T [R_K][R_J]^T \\ &\Rightarrow \boxed{\left[{}^J \dot{R}_K \right] = [{}^J \tilde{\omega}'_K]^T [{}^J R_K]} \quad (\text{components of } {}^J \omega_K \text{ are resolved in body } K) \end{aligned}$$