

ME 6590 Multibody Dynamics

Time Derivative of the (Coordinate) Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider a body $B:(e_1, e_2, e_3)$ moving in a fixed reference frame $R:(N_1, N_2, N_3)$. If r is a vector fixed in the body B , then the derivative of r may be written as

$$\boxed{\frac{{}^R dr}{dt} = \dot{r} = {}^R \omega_B \times r} \quad (1)$$

When performing the cross product, the individual vectors and the resulting cross product can be expressed in any reference frame. In the following paragraphs, “primes” indicate vector components in body $B:(e_1, e_2, e_3)$, and “no primes” indicate vector components in $R:(N_1, N_2, N_3)$. Also, let $[R]$ be the transformation matrix that relates the two sets of unit vectors as defined by the equation

$$\boxed{\{e\} = [R]\{N\}} \quad (2)$$

Case 1: \dot{r} expressed in $R:(N_1, N_2, N_3)$, but ${}^R \omega_B$ and r expressed in $B:(e_1, e_2, e_3)$

$$\text{In this case, write } \boxed{\dot{r} = \sum_{i=1}^3 \dot{r}_i N_i}, \quad \boxed{{}^R \omega_B = \sum_{i=1}^3 \omega'_i e_i}, \quad \text{and } \boxed{r = \sum_{i=1}^3 r'_i e_i}.$$

The three sets of vector components are related by the matrix form of Eq. (1).

$$\dot{r} = {}^R \omega_B \times r \quad \rightarrow \quad \boxed{\{\dot{r}\} = [R]^T ([\tilde{\omega}']\{r'\}) = ([R]^T [\tilde{\omega}'])\{r'\}} \quad (3)$$

Case 2: \dot{r} and ${}^R \omega_B$ expressed in $R:(N_1, N_2, N_3)$, but r expressed in $B:(e_1, e_2, e_3)$

$$\text{In this case, write } \boxed{\dot{r} = \sum_{i=1}^3 \dot{r}_i N_i}, \quad \boxed{{}^R \omega_B = \sum_{i=1}^3 \omega_i N_i}, \quad \text{and } \boxed{r = \sum_{i=1}^3 r'_i e_i}.$$

The three sets of vector components are again related by the matrix form of Eq. (1)

$$\dot{r} = {}^R \omega_B \times r \quad \rightarrow \quad \boxed{\{\dot{r}\} = [\tilde{\omega}][R]^T\{r'\} = ([\tilde{\omega}][R]^T)\{r'\}} \quad (4)$$

Time Derivative of the Transformation Matrices

The above results can be used to determine *two different forms* of the *time derivative* of the transformation matrix $[R]$. To do this, first note that the components of position vector \underline{r} in the two different reference frames are related as follows

$$\{r\} = [R]^T \{r'\}$$

This matrix equation can be *differentiated* directly to give

$$\boxed{\{\dot{r}\} = [\dot{R}]^T \{r'\} + [R]^T \{\dot{r}'\} = [\dot{R}]^T \{r'\}}$$

Here, advantage is taken of the fact that since \underline{r} is fixed in the body, $\boxed{\{\dot{r}'\} = \{0\}}$. Comparing this result with Eqs. (3) and (4) gives the two forms of $[\dot{R}]$.

$$\boxed{[\dot{R}_K]^T = [R_K]^T [\tilde{\omega}'_K]} \quad \text{and} \quad \boxed{[\dot{R}_K]^T = [\tilde{\omega}_K][R_K]^T}$$

or

$$\boxed{[\dot{R}_K] = [\tilde{\omega}'_K]^T [R_K]} \quad \text{and} \quad \boxed{[\dot{R}_K] = [R_K][\tilde{\omega}_K]^T}$$