

ME 3600 Control Systems

Examples: Using Laplace Transforms to Solve Differential Equations

Examples

1. Unforced Spring-Mass-Damper

Problem: Solve the differential equation of motion $m\ddot{x} + c\dot{x} + kx = 0$ subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

Solution: Taking Laplace transforms of both sides of the differential equation gives

$$m[s^2 X(s) - sx_0 - \dot{x}_0] + c[sX(s) - x_0] + kX(s) = 0$$

or

$$[ms^2 + cs + k]X(s) = [ms + c]x_0 + m\dot{x}_0$$

Solving for $X(s)$ gives

$$X(s) = \frac{[ms + c]x_0 + m\dot{x}_0}{ms^2 + cs + k} = \left(\frac{[ms + c]x_0}{ms^2 + cs + k} \right) + \left(\frac{m\dot{x}_0}{ms^2 + cs + k} \right)$$

Notes:

- The two terms on the right side of this equation represent the response of the system to the **initial position** and **initial velocity**, respectively.
- The **characteristic equation** of the system is found by setting the **denominator** of the right side of the equation to **zero** (i.e. $ms^2 + cs + k = 0$).
- The **poles** of the system are the roots of the **denominator**, and the **zeros** of the system are the roots of the **numerator**.

Case 1: $k/m = 2$; $c/m = 3$; $x_0 \neq 0$; $\dot{x}_0 = 0$

Substituting these values into the equation for $X(s)$ gives

$$X(s) = \frac{[ms + c]x_0}{ms^2 + cs + k} = \frac{x_0(s + 3)}{s^2 + 3s + 2} = \frac{x_0(s + 3)}{(s + 1)(s + 2)} \quad (2 \text{ real, unequal poles})$$

The solution to the differential equation may be found by taking the **inverse Laplace transform** of $X(s)$. Using #8 from the Laplace transform table with $\alpha = 3$, $a = 1$, and $b = 2$ gives

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left[\frac{x_0(s+3)}{(s+1)(s+2)}\right] = x_0(2e^{-t} - e^{-2t})$$

Check:

$$\begin{aligned} x(0) &= x_0(2e^0 - e^0) = x_0(2 - 1) = x_0 \\ \dot{x}(0) &= \dot{x}(t)\Big|_{t=0} = x_0(-2e^{-t} + 2e^{-2t})\Big|_{t=0} = x_0(-2 + 2) = 0 \end{aligned}$$

Case 2: $k/m = 2$; $c/m = 2$; $x_0 \neq 0$; $\dot{x}_0 = 0$

Substituting these values into the equation for $X(s)$ gives

$$X(s) = \frac{[ms + c]x_0}{ms^2 + cs + k} = \frac{x_0(s+2)}{s^2 + 2s + 2} = \frac{x_0(s+2)}{(s+1)^2 + 1} \quad (2 \text{ complex conjugate poles})$$

The solution to the differential equation may be found by taking the *inverse Laplace transform* of $X(s)$. Using #18 from the Laplace transform table with $\alpha = 2$, $a = 1$, and $\omega = 1$ gives

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left[\frac{x_0(s+2)}{(s+1)^2 + 1}\right] = \sqrt{2} x_0 e^{-t} \sin(t + \phi)$$

where $\phi = \tan^{-1}(1/(2-1)) = \begin{cases} 0.7854 \text{ (rad)} = 45 \text{ (deg)} \\ 0.7854 + \pi \text{ (rad)} = 225 \text{ (deg)} \end{cases}$.

Check: (using $\phi = 0.7854$ (rad))

$$\begin{aligned} x(0) &= x(t)\Big|_{t=0} = \sqrt{2} x_0 e^0 \sin(0.7854) = \sqrt{2} x_0 \left(\frac{\sqrt{2}}{2}\right) = x_0 \\ \dot{x}(0) &= \dot{x}(t)\Big|_{t=0} = \sqrt{2} x_0 \left[-e^{-t} \sin(t + 0.7854) + e^{-t} \cos(t + 0.7854)\right]\Big|_{t=0} \\ &= \sqrt{2} x_0 \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 0 \end{aligned}$$

Note that $\phi = 0.7854 + \pi$ (rad) **does not** satisfy the initial conditions.

2. Spring-Mass-Damper with a *Unit Step Input*

Problem: Solve the differential equation of motion $m\ddot{x} + c\dot{x} + kx = R(t) = u_s(t)$ where $u_s(t)$ is the *unit step function*. Find the *final value* of $x(t)$ using the *final value theorem*.

Case 1: $m = 1$; $k/m = 2$; $c/m = 3$; $x_0 = \dot{x}_0 = 0$

Taking Laplace transforms of both sides of the differential equation gives

$$(ms^2 + cs + k)X(s) = \mathcal{L}(u_s(t)) = \frac{1}{s} \quad \text{or} \quad \boxed{X(s) = \frac{1}{s(s+1)(s+2)}}$$

Using #9 from the Laplace transform tables with $a = 1$ and $b = 2$ gives

$$\boxed{x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{2}[1 - 2e^{-t} + e^{-2t}]} \quad (\text{forced response})$$

Using the final value theorem, we have $x_{ss} = \lim_{s \rightarrow 0} (sX(s)) = \lim_{s \rightarrow 0} \left(\frac{\cancel{s}}{\cancel{s}(s+1)(s+2)} \right) = \frac{1}{2}$.

Case 2: $m = 1$; $k/m = 2$; $c/m = 3$; $x_0 \neq 0$; $\dot{x}_0 = 0$

Taking Laplace transforms of both sides of the differential equation gives

$$m[s^2 X(s) - sx_0 - \cancel{\dot{x}_0}] + c[sX(s) - x_0] + kX(s) = \frac{1}{s}$$

or

$$[ms^2 + cs + k]X(s) = \frac{1}{s} + [ms + c]x_0$$

Solving for $X(s)$ gives

$$\boxed{X(s) = \left(\frac{1}{s(ms^2 + cs + k)} \right) + \left(\frac{x_0(ms + c)}{ms^2 + cs + k} \right) = \underbrace{\left(\frac{1}{s(s^2 + 3s + 2)} \right)}_{\text{forced response}} + \underbrace{\left(\frac{x_0(s + 3)}{(s^2 + 3s + 2)} \right)}_{\text{response due to initial condition}}}$$

Using #8 and #9 from the Laplace transform tables gives

$$\boxed{x(t) = \mathcal{L}^{-1}(X(s)) = \frac{1}{2} \underbrace{[1 - 2e^{-t} + e^{-2t}]}_{\text{forced response}} + x_0 \underbrace{[2e^{-t} - e^{-2t}]}_{\text{response due to initial condition}}}$$

Question: What part of this response is *transient response* and what part is *steady-state response*?