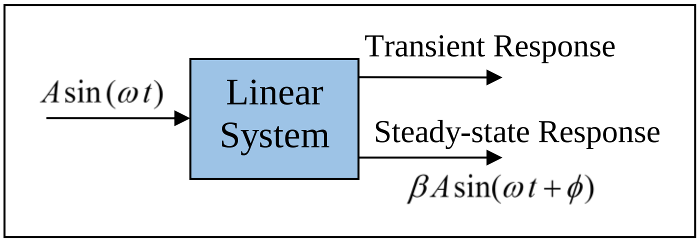
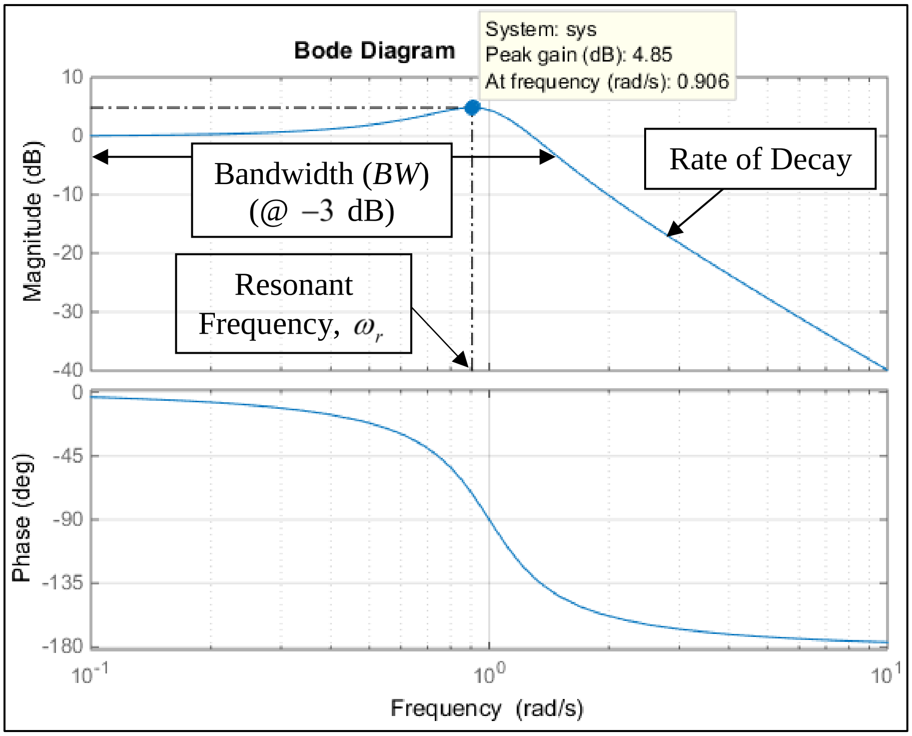
ME 3600 Control Systems
Frequency Domain Analysis

The ***frequency response*** of a system is defined as the ***steady-state response*** of the system to a ***sinusoidal (harmonic) input***. For ***linear*** systems, the resulting steady-state output is itself harmonic; it differs from the input in ***amplitude*** and ***phase*** only.



Here, β represents the ***multiplication factor*** for the ***magnitude***, and ϕ represents the ***relative phase shift*** between the input and the output. If $\beta > 1$, the system ***amplifies*** the input, and if $\beta < 1$, the system ***attenuates*** the input.

One common way to represent the frequency response of a linear system is using a ***Bode diagram***. The Bode diagram of a ***typical second order system*** is shown in the diagram below. Using the magnitude plot, the resonant frequency (ω_r) the resonant magnitude (M_r), the bandwidth (*BW*), and the ***rate of decay*** of the system after resonance can be identified.



The Bode diagram shown above is for a second-order system whose natural frequency is $\omega_n = 1$ (rad/s) and whose damping ratio is $\zeta = 0.3$. The resonant frequency ω_r is defined as the frequency at which the magnitude is maximum. The magnitude at this frequency is M_r. In this case, MATLAB® indicates the resonant frequency and resonant magnitude are $\omega_r = 0.906$ (rad/s) and $20\log(M_r) = 4.85$ (dB). The bandwidth (*BW*) of the system is defined as the frequency at which the system is 3 dB ***down*** from its constant low-frequency value. Note that -3 dB represents an amplitude multiplier of 0.71, so the system reponds at 71% of its low frequency value. The decay rate after resonance for a second-order system is -40 (dB/decade).

If a system is second-order and has a damping ratio $\zeta < 0.707$, ***the resonant frequency*** and ***resonant magnitude*** may be ***estimated*** using the equations

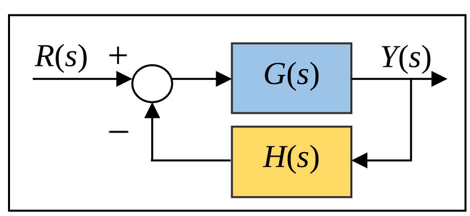
$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}} \quad \text{and} \quad \boxed{M_r = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}} \qquad (1)$$

Eqs. (1) can be used for ***any pair*** of ***complex poles*** (in higher-order systems) that are ***sufficiently isolated*** (in frequency) from other poles and zeros. In these cases, M_r represents the ***rise*** in magnitude from the ***pre-resonance value***.

These ***characteristics*** in the ***frequency-domain correlate*** with ***behavior*** of the system in the ***time-domain***. The resonant magnitude M_r gives an indication of the ***relative stability***. ***Large values*** of M_r are indicative of ***low damping***, suggesting ***oscillatory*** response with potentially ***large overshoots***. Systems with ***large bandwidths*** have ***faster response*** than systems with small bandwidths; however, they may be more ***noise sensitive***. Sensitivity to noise is determined by a ***combination*** of the ***bandwidth*** and the ***rate of decay*** of the magnitude at high frequencies.

Minimum Phase Systems

The loop transfer function $GH(s)$ of a ***minimum phase system*** has ***no zeros*** or ***poles in the right-half of the s-plane***. If a system has poles or zeros in the right-half plane, it is referred to as a ***non-minimum phase system***.



Simple Closed Loop System

If a closed-loop system is a ***minimum phase*** system, then the ***stability*** of the system can be determined by examining the Bode diagram of the loop transfer function $GH(s)$. If the system is a ***non-minimum phase*** system, a ***Nyquist diagram*** can be used to determine stability. Bode diagrams are usually preferred over Nyquist diagrams for minimum phase systems, because it is easier to measure the gain and phase margins on a Bode diagram. It is also easier to see how the Bode diagram changes shape as poles and zeros are added to (or removed from) the system. The Nyquist diagram and the Nyquist stability criterion are presented below.

Gain and Phase Margins and the Bode Diagram

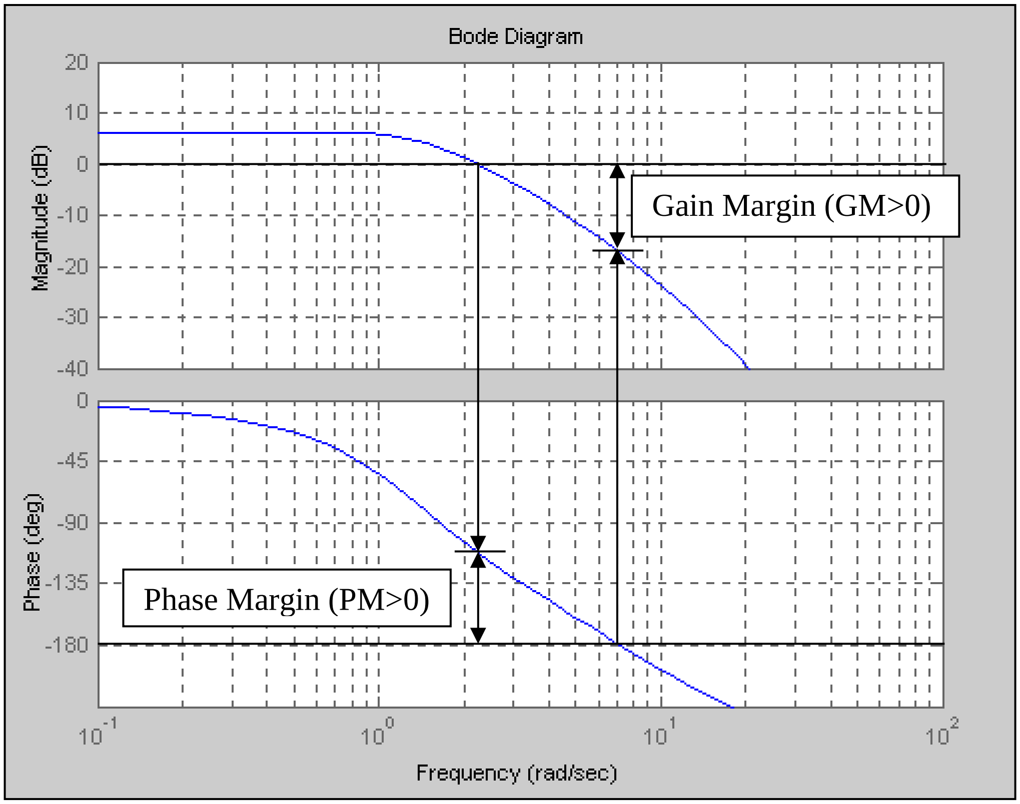
Gain and ***phase margins*** of a ***minimum phase*** system are determined by plotting the Bode diagram of the loop transfer function $GH(s)$. To illustrate this process, consider the Bode diagram of the loop transfer function

$$\boxed{GH(s) = \frac{100(s+2)}{(s+5)(s+10)(s^2+2s+2)}} \qquad (2)$$

Phase margin (*PM*) is the ***additional phase lag*** required to make the phase angle $-180\ (\text{deg})$ at the frequency where the magnitude of the system crosses the ***zero-dB*** line. ***Gain margin*** (*GM*) is the ***additional magnitude*** required to make the magnitude ***zero*** dB when the phase angle is $-180\ (\text{deg})$. In the case shown below, the ***phase margin*** is $PM = +68\ (\text{deg})$ (measured at 2.2 (rad/s)), and the ***gain margin*** is $GM = +17.4\ \text{dB}$ (measured at 7.15 (rad/s)).

Gain and Phase Margins and Stability

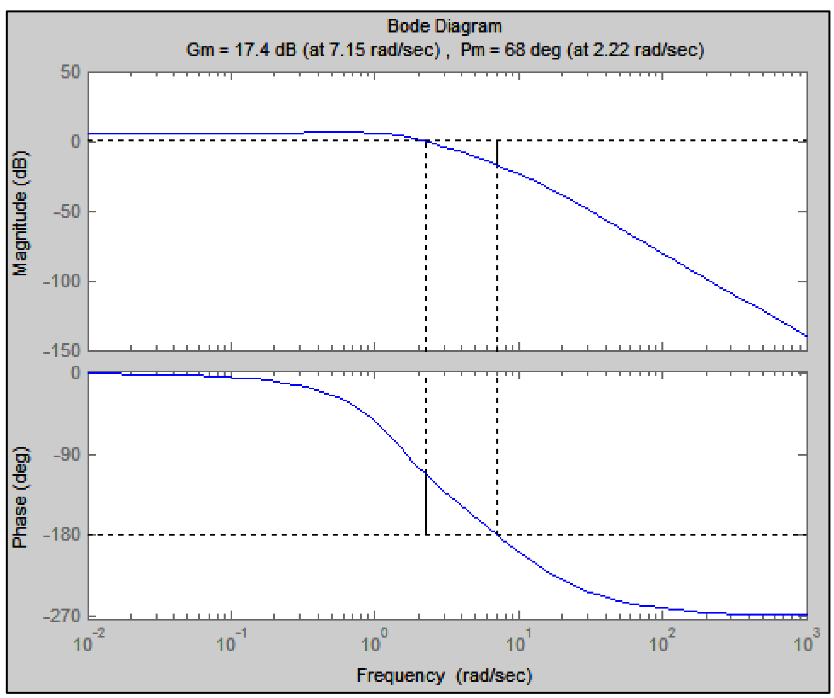
The ***gain*** and ***phase margins*** determine the ***stability*** of ***minimum phase*** systems. A ***minimum phase system*** is ***stable*** if ***both margins*** are ***positive***, and ***unstable*** if they are ***negative***. Systems with a ***higher degree of stability*** have ***larger margins*** and ***less stable*** systems have ***smaller margins***. The Bode diagram below represents a ***stable*** closed-loop system.



MATLAB Commands for Bode Diagrams and Gain and Phase Margins

A set of MATLAB® commands to display the ***Bode diagram*** and the ***gain*** and ***phase margins*** for the transfer function of Eq. (2) are shown below. The figure shows the display resulting from the “margin” command. Note that the convolution function “`conv`” is used to build the transfer function from its component parts.

```
>> num = 100*[1,2];

>> den = conv([1,5],conv([1,10],[1,2,2]));

>> sys = tf(num, den)

   Transfer function:
                100 s + 200
     ------------------------------------
   s^4 + 17 s^3 + 82 s^2 + 130 s + 100

>> bode(sys); grid;

>> margin(sys)
```

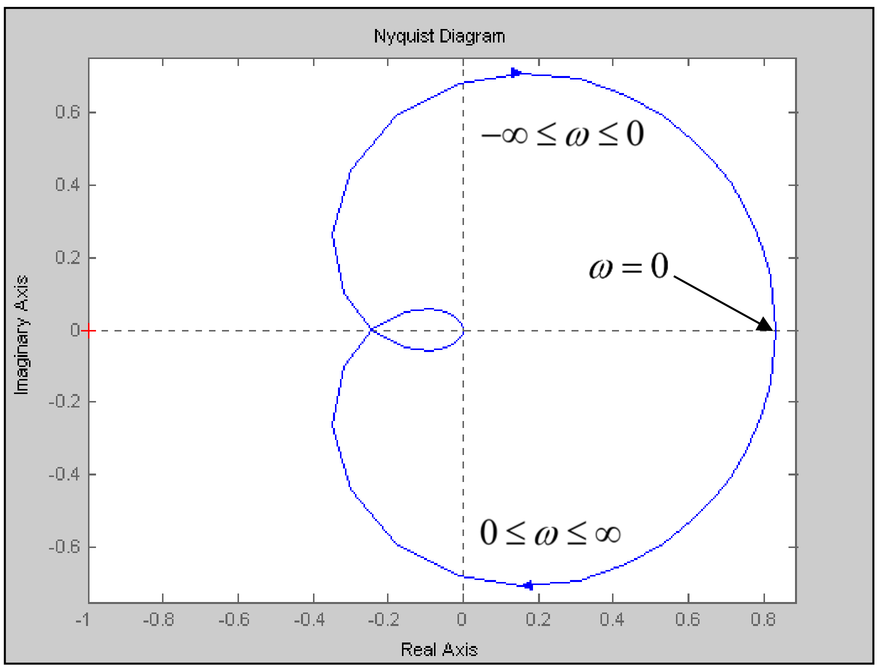


Nyquist Diagram

A Nyquist diagram is a plot of the ***real-part*** of $GH(j\omega)$ versus the ***imaginary-part*** of $GH(j\omega)$. (Recall that the Bode diagram is a plot of the magnitude of $GH(j\omega)$ versus ω and a plot of the phase of $GH(j\omega)$ versus ω.) The Nyquist diagram for the loop transfer function

$$\boxed{GH(s) = \frac{(s+10)}{(s+2)(s+3)(s^2+2s+2)}} \quad (3)$$

is shown in the diagram below. Note that $GH(j\omega)\big|_{\omega=0} = 0.833 + j0$, and the arrow heads show the directions of ***increasing frequency***. The part of the diagram that is ***below*** the ***real axis*** is for the frequency range $0 \le \omega \le \infty$, and the part ***above*** the ***real axis*** (the complex conjugate of the part below) is for the range $-\infty \le \omega \le 0$. Note also that the plot forms a ***closed curve***.



Before stating the ***Nyquist stability criterion***, a few definitions are helpful.

N = number of times the Nyquist plot ***encircles*** the point $-1+j0$ in a ***clockwise sense*** ($-1+j0$ is denoted on the diagram with a red "+")

Z = number of ***zeros*** of $1+GH(s)$ in the ***right-half*** of the *s*-plane

P = number of ***poles*** of $1+GH(s)$ in the ***right-half*** of the *s*-plane (Note that the poles of $1+GH(s)$ are also the poles of $GH(s)$)

Nyquist Stability Criterion

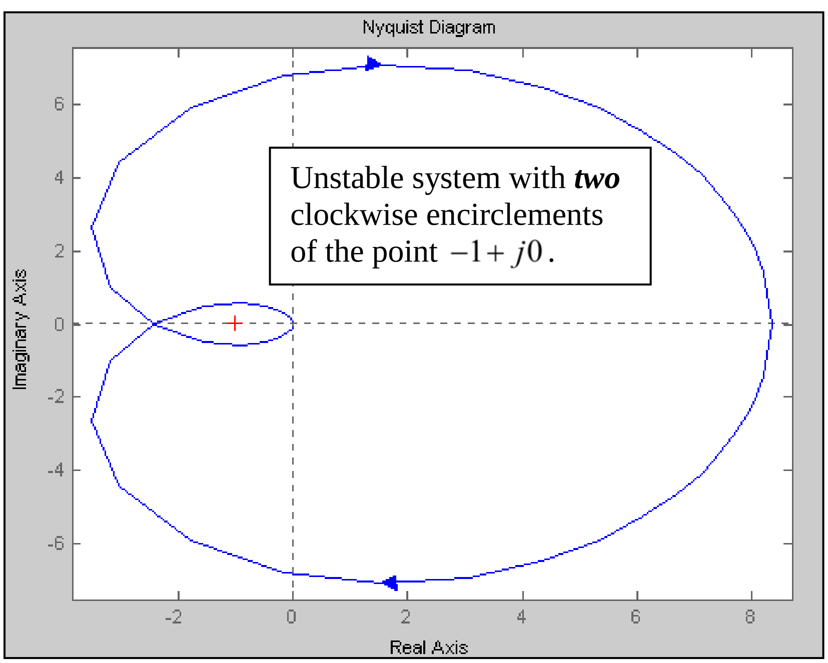
For a closed loop system to be ***stable*** the Nyquist criterion states that $Z=0$ and $N=Z-P=-P$. This means that the Nyquist plot must encircle the point $-1+j0$ "P" times in the ***counterclockwise*** direction for the closed-loop system to be stable. Note that if the loop transfer function $GH(s)$ is of ***minimum phase***, then $P=0$ and $N=0$. Hence, the Nyquist plot for a ***stable***, ***minimum-phase*** system ***does not encircle*** the point $-1+j0$. The Nyquist diagram above indicates the closed loop system is ***stable***.

If the closed-loop system is ***unstable***, the Nyquist criterion states that the number of poles of the closed-loop system (also the number of zeros of $1+GH(s)$) in the ***right-half*** plane is equal to $Z=N+P$. If the system is ***minimum phase***, the number of poles in the right-half plane is equal to the number of ***clockwise*** encirclements of the point $-1+j0$.

To illustrate this last statement, consider the Nyquist plot for the loop transfer function in Eq. (4) as shown in the figure below.

$$GH(s)=\frac{10\,(s+10)}{(s+2)(s+3)(s^2+2s+2)} \tag{4}$$

The Nyquist plot encircles the point $-1+j0$ ***twice*** in a ***clockwise direction***. This means that the closed-loop system has ***two poles*** in the ***right-half*** plane, indicating the system is ***unstable***.

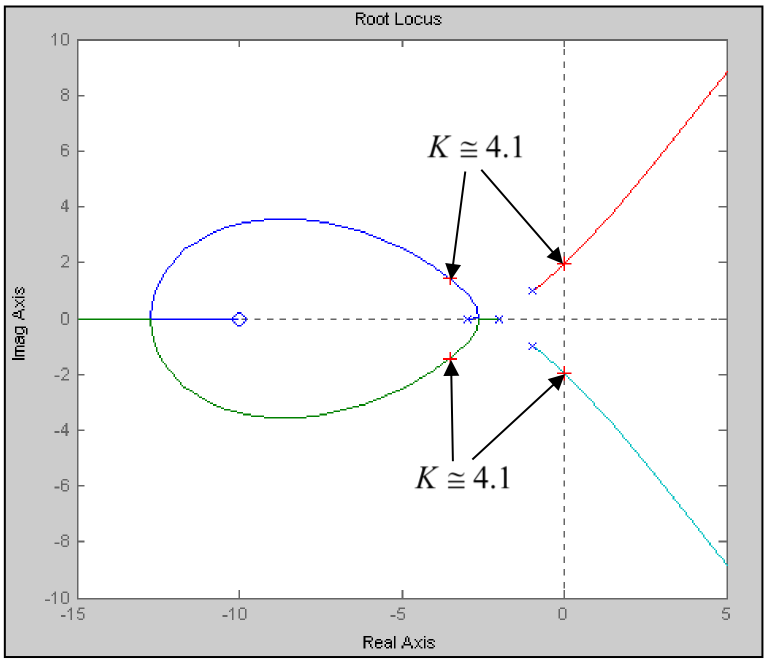


Comparison of Results from a Root Locus Diagram, a Bode Diagram, and a Nyquist Diagram

To compare the results obtained from root locus, Bode, and Nyquist diagrams for a single system. Consider the closed-loop system whose loop transfer function is

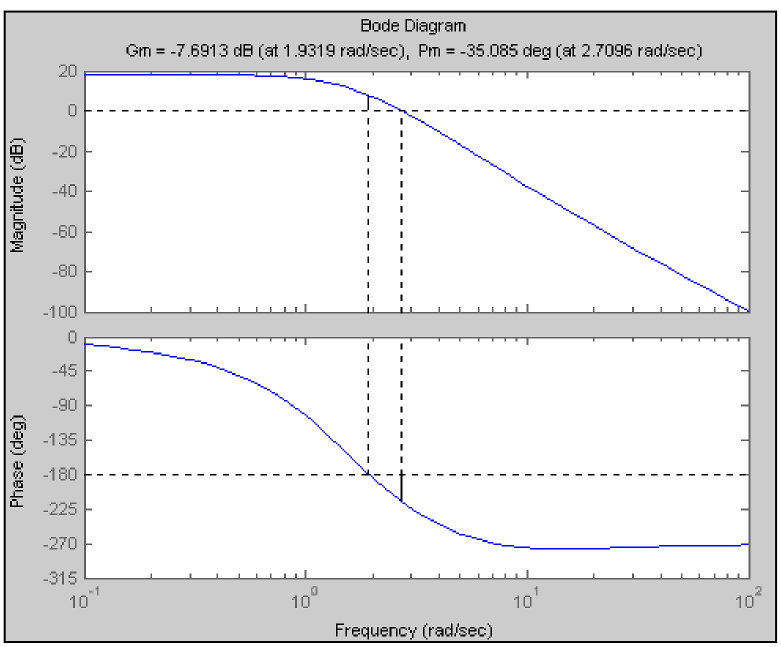
$$GH(s) = \frac{K\,(s+10)}{(s+2)(s+3)(s^2+2s+2)} \qquad (5)$$

The first diagram below shows the root locus diagram for the parameter K indicating the system is ***unstable*** when $K > 4.1$. The next two diagrams are the Bode and Nyquist diagrams for the system with $K = 10$. It is clear from these diagrams that the system is ***unstable*** for $K = 10$.



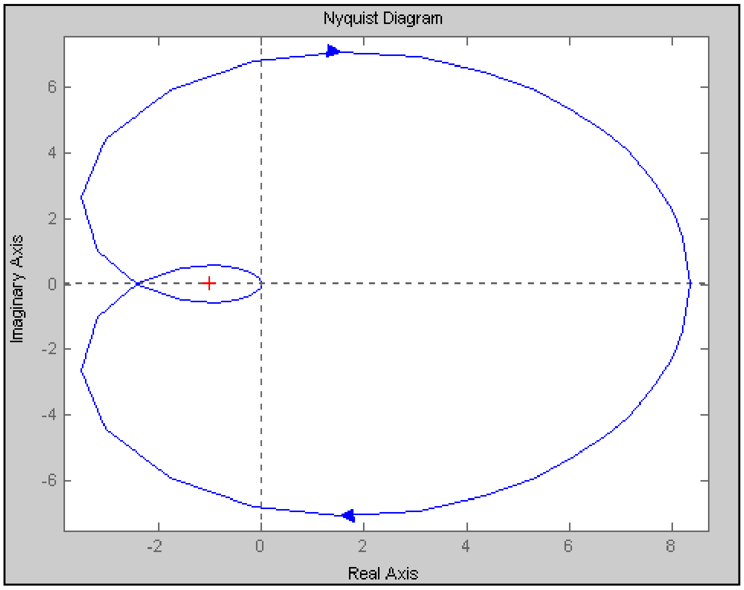
Root Locus Diagram

System is ***unstable*** for $K > 4.1$



Bode Diagram

Verifies the system is ***unstable*** for $K = 10$.
($GM = -7.7$ (dB) and $PM = -35$ (deg))



Nyquist Diagram

Verifies the system is ***unstable*** for $K = 10$.
(Plot encircles $-1 + j0$ twice in a clockwise direction, indicating two closed-loop poles in the right half *s*-plane.)