

## ME 3600 Control Systems

### Introduction to Root Locus Diagrams

The *dynamic characteristics* (transient response and stability) of a linear system are dependent on the *location* of the *poles* of the system. When a compensator (controller) is used to provide closed-loop control of a system, the *poles* of the *closed-loop transfer function* determine the system's dynamic characteristics. The *location* of these poles are *functions* of the compensator's parameters. The root locus method provides a means to *track* the locations of the poles of the closed-loop system as a function of a *single parameter* of the system or compensator.

A *root locus* is the *path* of a pole in the  $s$ -plane (complex plane) as a function of a single parameter. A *root locus diagram* provides the *paths* of *all poles* of the system as a function of that parameter. Using this method, an analyst can choose values of the parameter that will yield *acceptable* dynamic characteristics.

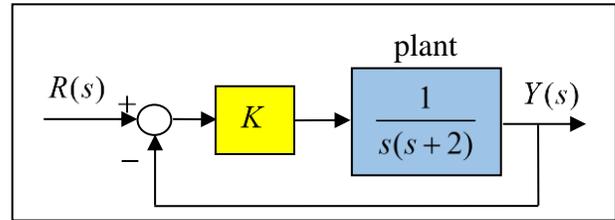
A *root locus diagram* can be *generated* from *graphical considerations* or by using *computer software*. Clearly a design engineer will prefer to generate the details of these diagrams using a computer program but understanding of the graphical methods can provide the analyst with a better *understanding* of *how* the general characteristics of a root locus diagram will change as the type of compensator is altered. With this motivation, the notes that follow will discuss both approaches.

#### Root Locus Diagrams – Basic Numerical Method

As shown in the notes entitled “What is a Root Locus Diagram?”, the most *basic method* for generating a root locus diagram is to find the *characteristic equation* of the system as a *function* of some parameter, say  $K$ , and then proceed to *calculate the roots* of the equation for a series of values of  $K$ . The roots are then plotted to form the root locus diagram. The *movement* of the poles are *easily tracked*, but *little information* is available about how the diagram will change as the form of the controller is altered.

## Root Locus Diagrams – Basic Graphical Method

To illustrate the *basic graphical method*, consider the simple closed-loop system shown in the diagram. The closed-loop transfer function of the system is



$$\frac{Y}{R}(s) = \frac{G}{1 + GH(s)} = \frac{\frac{K}{s(s+2)}}{1 + \frac{K}{s(s+2)}} = \frac{K}{s(s+2) + K}$$

The *characteristic equation* of the system can then be written as

$$1 + K \underbrace{\left( \frac{1}{s(s+2)} \right)}_{P(s)} = 0 \quad \text{or} \quad s(s+2) + K = 0 \quad (1)$$

As will be discussed in subsequent notes, the first of these is considered the *standard form* for root locus diagrams. The polynomial ratio  $P(s)$  is defined as the multiplier of the parameter  $K$  in the characteristic equation (Eq. (1)). Generally, both the numerator and denominator of  $P(s)$  will be polynomials.

To generate the root locus diagram ( $K \geq 0$ ) for this system, first rewrite the second form of the characteristic equation as follows

$$s(s+2) = -K$$

Now replace “s” and “s+2” in this equation with the polar forms (recall that  $j = \sqrt{-1}$ )

$$s = a_1 e^{j\alpha_1} \quad \text{and} \quad s+2 = a_2 e^{j\alpha_2}$$

Substituting these forms into the characteristic equation gives

$$a_1 a_2 e^{j(\alpha_1 + \alpha_2)} = -K + 0j = K e^{jn\pi}$$

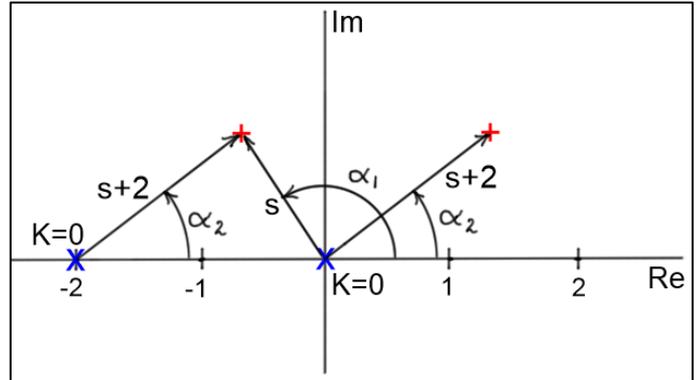
This equation provides the following two scalar equations

$$a_1 a_2 = K \quad \text{and} \quad \alpha_1 + \alpha_2 = n\pi \quad (n = \pm 1, \pm 2, \dots) \quad (2)$$

So, if a point in the  $s$ -plane is a root of the characteristic equation, it must satisfy these two equations. The paths of the poles start with  $K = 0$  and proceed as  $K \rightarrow \infty$ . Note from the characteristic equation that if  $K = 0$ ,  $s = 0$  and  $s = -2$ .

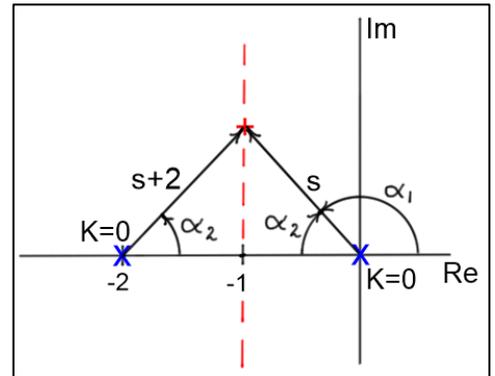
Note on Geometry in the  $s$ -Plane:

The diagram to the right shows a geometric representation of the terms “ $s$ ” and “ $s + 2$ ” in the  $s$ -plane. The vectors are shown with both tails at the origin and with both ends at “ $s$ ”. When both ends are at “ $s$ ”, the tails start at the poles of  $P(s)$  at  $s = 0$  and  $s = -2$ .

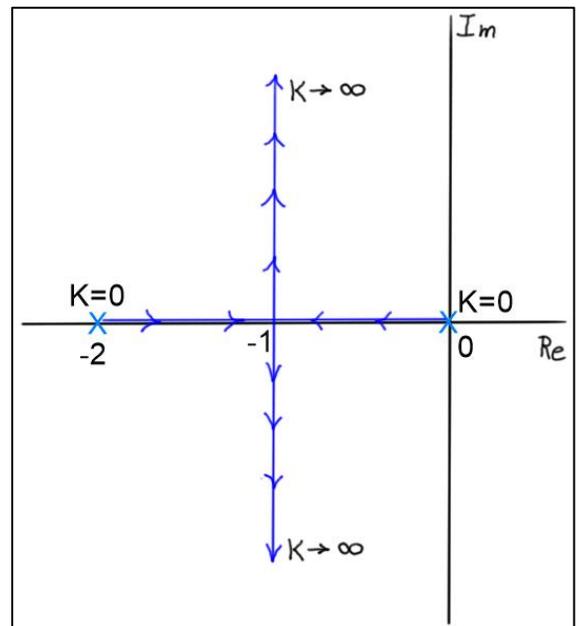


Locations on the Root Locus Diagram

For a point to be on the root locus diagram of the example system shown above, the second of Eqs. (2) requires the sum of the angles  $\alpha_1$  and  $\alpha_2$  must be a multiple of  $\pi$  (or 180 degrees). The diagram to the right shows that all points on the dashed red line crossing the real axis at  $s = -1$  satisfy this criterion.



At any point on the dashed red line that is off the real axis, the vectors for “ $s$ ” and “ $s + 2$ ” form an isosceles triangle as shown. Both angles are non-zero, but their sum is  $\pi$ . Note also that any point on the real axis in the range  $-2 < s < 0$  also satisfies the angle criterion. At these points  $\alpha_1 = \pi$  and  $\alpha_2 = 0$ . Note finally that the value of  $K$  increases as the points move further away from the starting points at  $s = 0$  and  $s = -2$ . The final root locus diagram has two branches and is shown in the diagram to the right.



### Values of $K$ on the Two Branches

The value of  $K$  at any point on the branches can be calculated using the first of Eqs. (2).

The values in following table were found using this approach.

Location	$s$	$s + 2$	$a_1$	$a_2$	$K$
$-0.5 + 0j$	$-0.5 + 0j$	$1.5 + 0j$	0.5	1.5	0.75
$-1.5 + 0j$	$-1.5 + 0j$	$0.5 + 0j$	1.5	0.5	0.75
$-1 + 0j$	$-1 + 0j$	$1 + 0j$	1	1	1
$-1 + 1j$	$-1 + 1j$	$1 + 1j$	$\sqrt{2}$	$\sqrt{2}$	2
$-1 - 1j$	$-1 - 1j$	$1 - 1j$	$\sqrt{2}$	$\sqrt{2}$	2

The table shows the system has *two real, unequal poles* for  $0 < K < 1$ , *two real, equal poles* for  $K = 1$ , and a *pair of complex conjugate poles* for  $K > 1$ . Hence, the closed-loop system is *over-damped* for  $0 < K < 1$ , *critically damped* for  $K = 1$ , and *under-damped* for  $K > 1$ .

### Closure

Following the above procedure for *more complex systems* would be *very tedious* and *time consuming*. Fortunately, there is a *much faster, less detailed approach* that can be taken. That approach is *presented* and is *applied* to a few example systems in the notes that follow. Recall the *motivation* for sketching a root locus diagram is for the analyst to have a *clearer understanding* of the general impact various forms of control will have on a system. Once a candidate method is found, more *detailed root locus diagrams* can be generated using *computer software*.