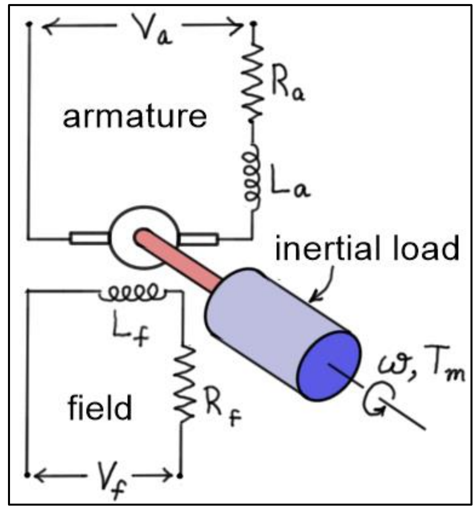
ME 4710 Motion and Control
Armature Controlled DC Motor Transfer Functions
(Reference: Dorf and Bishop, Modern Control Systems, 10th Ed., Pearson/Prentice-Hall, Inc. 2005)

In an ***armature-current controlled DC motor***, the field current i_f is held constant, and the armature current is controlled through the armature voltage V_a. In this case, the ***motor torque increases linearly*** with the ***armature current***.



DC Motor

$$\boxed{T_m = K_{ma}\, i_a} \qquad (1)$$

The coefficient K_{ma} is a ***constant*** that depends on the chosen motor. The ***transfer function*** from the ***input armature current*** to the ***resulting motor torque*** is

$$\boxed{\frac{T_m}{I_a}(s) = K_{ma}} \qquad (2)$$

The ***voltage-current relationship*** for the armature side of the motor is

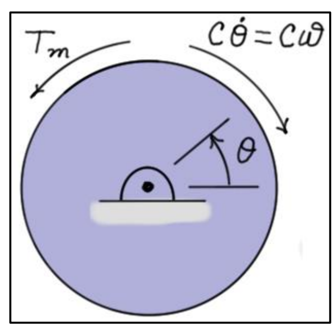
$$\boxed{V_a = V_R + V_L + V_b = R_a i_a + L_a\left(di_a/dt\right) + V_b} \qquad (3)$$

V_b represents the "***back EMF***" ***induced*** by the ***rotation*** of the ***armature windings*** in a ***magnetic field***. V_b is ***proportional*** to the ***rotational speed*** ω, that is, $\boxed{V_b(s) = K_b\omega(s)}$.

Taking Laplace transforms of Eq. (3) gives

$$\boxed{V_a(s) - V_b(s) = \left(R_a + L_a s\right) I_a(s)} \quad \text{or} \quad \boxed{V_a(s) - K_b\omega(s) = \left(R_a + L_a s\right) I_a(s)} \qquad (4)$$

Applying ***Newton's second law*** (by summing moments) for the rotational motion of the motor gives



Free Body Diagram
of the Inertial Load

$$\boxed{\sum M = T_m - c\omega = J\dot{\omega}} \qquad \text{(CCW positive)}$$

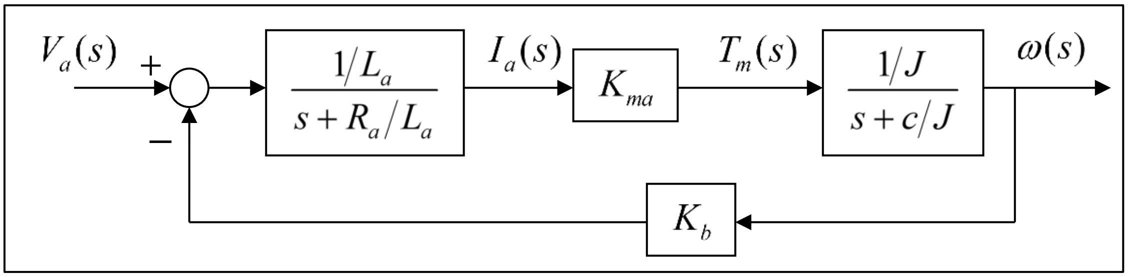
or

$$\boxed{J\dot{\omega} + c\omega = T_m} \qquad (5)$$

Taking Laplace transforms of both sides of Eq. (5) gives the ***transfer function*** from the ***input motor torque*** to ***rotational speed changes***.

$$\boxed{\frac{\omega}{T_m}(s) = \frac{(1/J)}{s+(c/J)}} \qquad \text{(1st order system)} \qquad (6)$$

Together, Eqs. (2), (4) and (6) can be represented by the following ***closed-loop block diagram***.



Using ***block diagram reduction***, the transfer function from the ***input armature voltage*** to the ***resulting speed change*** is found to be

$$\boxed{\frac{\omega}{V_a}(s) = \frac{(K_{ma}/L_a J)}{(s+R_a/L_a)(s+c/J)+(K_b K_{ma}/L_a J)}} \qquad \text{(2nd order system)} \qquad (7)$$

If the ***time constant*** of the ***electrical circuit*** is ***small*** compared to the ***time constant*** of the ***load dynamics***, the transfer function of Eq. (7) may be approximated by the following first-order transfer function.

$$\boxed{\frac{\omega}{V_a}(s) = \frac{K_{ma}/R_a J}{s+(cR_a+K_b K_{ma})/R_a J}} \qquad \text{(1st order system)} \qquad (8)$$

The transfer function from the ***input armature voltage*** to the resulting ***angular position*** change is found by multiplying Eqs. (7) and (8) by $1/s$.

$$\boxed{\frac{\theta}{V_a}(s) = \frac{K_{ma}/R_a J}{s\left(s+(cR_a+K_b K_{ma})/R_a J\right)}} \qquad \text{(2nd order system)} \qquad (9)$$

Note that this transfer function also represents a second-order differential equation with inertia and damping, but no stiffness (same form as for a hydraulic cylinder!).