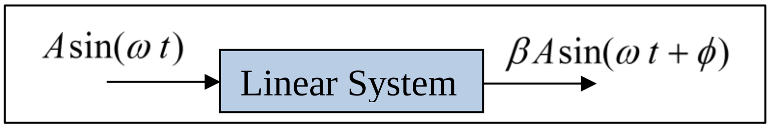
ME 4710 Motion and Control
Frequency Domain Analysis

The ***frequency response*** of a system is defined as the ***steady-state response*** of the system to a ***sinusoidal (harmonic) input***. For ***linear*** systems, the resulting output is itself harmonic; it differs from the input in ***amplitude*** and ***phase*** only. This is illustrated in the following block diagram.

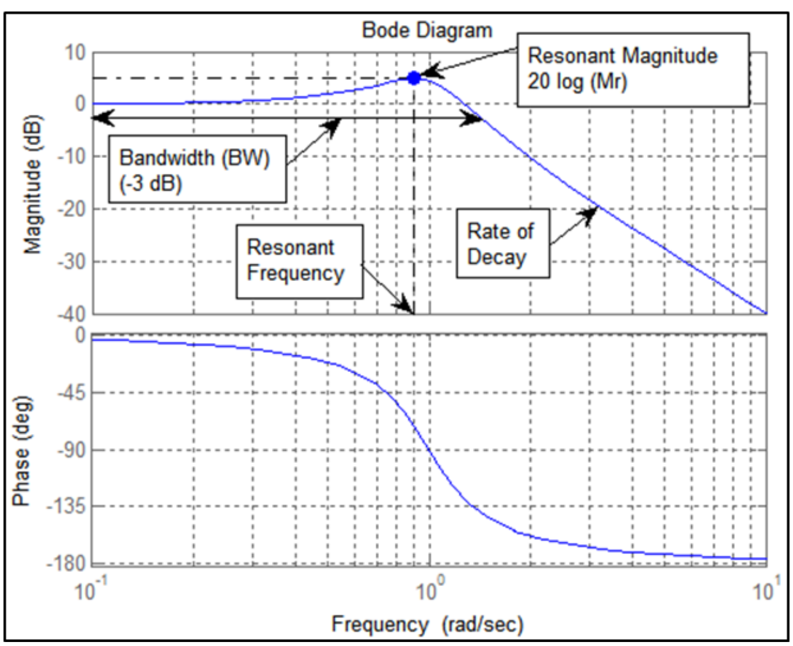


Steady-State Response

Here, β represents the ***multiplication factor*** for the magnitude, and ϕ represents the ***relative phase shift*** between the input and the output. If $\beta > 1$, the system ***amplifies*** the input, and if $\beta < 1$, the system ***attenuates*** the input.

One common way to represent the frequency response of a linear system is using a ***Bode diagram***. The Bode diagram of a ***typical second-order system*** is shown in the diagram below. Using the magnitude plot, the ***resonant frequency*** (ω_r) the ***resonant magnitude*** (M_r), the ***bandwidth*** (*BW*), and the ***rate of decay*** of the system response can be identified.

The bandwidth (*BW*) is defined as the frequency at which the system is 3dB down from its low frequency value. Note that –3*dB* represents an amplitude of 0.71, so the system reponds at 71% of its low frequency value.



If a system is ***second-order*** and has a damping ratio $\zeta < 0.707$, ***the resonant frequency*** and ***resonant magnitude*** can be ***estimated*** using the equations

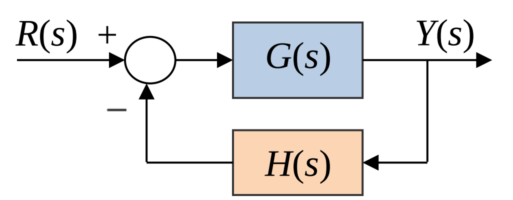
$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}} \quad \text{and} \quad \boxed{M_r = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}} \tag{1}$$

Note: Eqs. (1) can be used for ***any pair*** of complex poles (in higher order systems) that are ***sufficiently isolated*** (in frequency) from other poles and zeros. In this case, M_r represents the ***rise*** in magnitude from the ***pre-resonance value***.

These ***characteristics*** in the ***frequency-domain correlate*** with ***behavior*** of the system in the ***time-domain***.

- The ***resonant magnitude*** M_r gives an indication of the ***relative stability***. Large values of M_r are indicative of low damping, suggesting oscillatory response with large overshoots.
- Systems with ***large bandwidths*** have ***faster response*** than systems with small bandwidths; however, they may be more sensitive to high frequency noise.
- Sensitivity to noise is determined by a combination of the ***bandwidth*** and the ***rate of decay*** of the magnitude at high frequencies.

<u>Minimum Phase Systems</u>



Closed Loop System

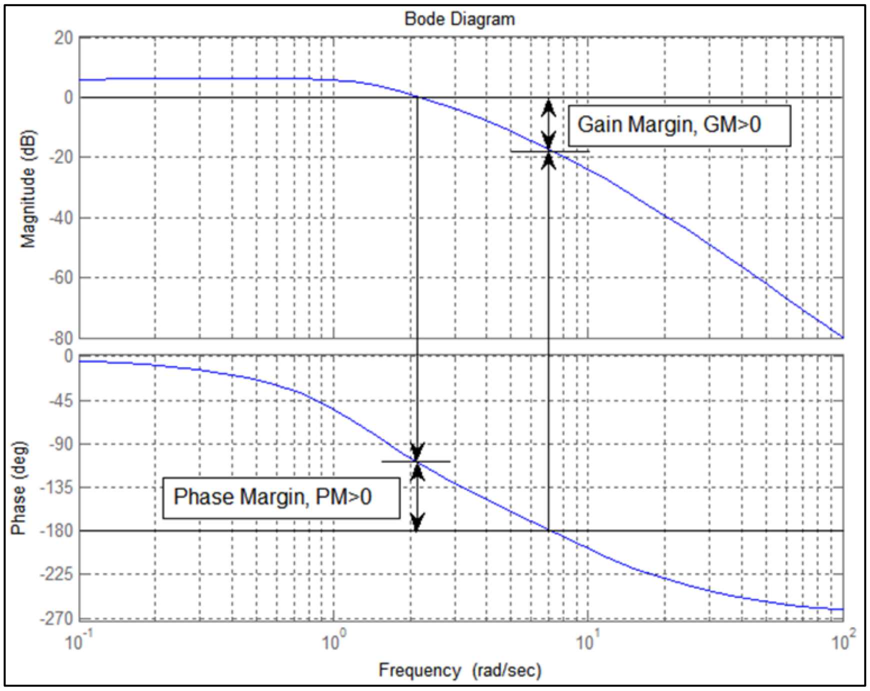
- The loop transfer function $GH(s)$ of a ***minimum phase system*** has ***no zeros or poles*** in the ***right half*** of the ***s-plane***.
- If a system has poles or zeros in the right-half plane, it is referred to as a ***non-minimum phase system***.
- If closed-loop system is a ***minimum phase*** system, then the ***stability*** of the system can be determined by examining the Bode diagram of the ***loop transfer function*** $GH(s)$.
- If the system is a ***non-minimum phase*** system, a ***Nyquist diagram*** can be used to determine stability.
- Bode diagrams are usually preferred over Nyquist diagrams for minimum phase systems, because it is easier to measure the gain and phase margins on a Bode diagram. It is also easier to see how the Bode diagram changes shape as poles and zeros are added to (or removed from) the system.

Gain and Phase Margins and the Bode Diagram

Gain and ***phase margins*** of a ***minimum phase*** system are determined by plotting the Bode diagram of the loop transfer function $GH(s)$. For example, consider the Bode diagram of the loop transfer function

$$\boxed{GH(s)=\frac{100(s+2)}{(s+5)(s+10)(s^2+2s+2)}} \tag{2}$$

The ***phase margin*** (*PM*) is the ***additional phase lag*** required to make the phase angle $-180\ \text{(deg)}$ at the frequency where the magnitude of the system crosses the ***zero-dB*** line. The ***gain margin*** (*GM*) is the ***additional magnitude*** required to make the magnitude ***zero dB*** when the ***phase angle*** is $-180\ \text{(deg)}$. In the case shown below, the ***phase margin*** is $PM=+68\ \text{(deg)}$ (measured at 2.2 (rad/s)), and the ***gain margin*** is $GM=+17.4\ \text{dB}$ (measured at 7.15 (rad/s)).



Gain and Phase Margins and Stability

- The ***gain*** and ***phase margins*** determine the ***stability*** of ***minimum phase*** systems. A ***minimum phase system*** is ***stable if both margins*** are ***positive***, and ***unstable*** if they are ***negative***.
- Systems with a ***higher degree of stability*** have ***larger margins*** and ***less stable*** systems have ***smaller margins***. The Bode diagram above represents a ***stable*** closed-loop system.

MATLAB Commands for Bode Diagrams and Gain and Phase Margins

A set of MATLAB commands to display the ***Bode diagram*** and the ***gain*** and ***phase margins*** for the transfer function of Eq. (2) are shown below. The figure shows the display resulting from the "margin" command.

MATLAB Commands

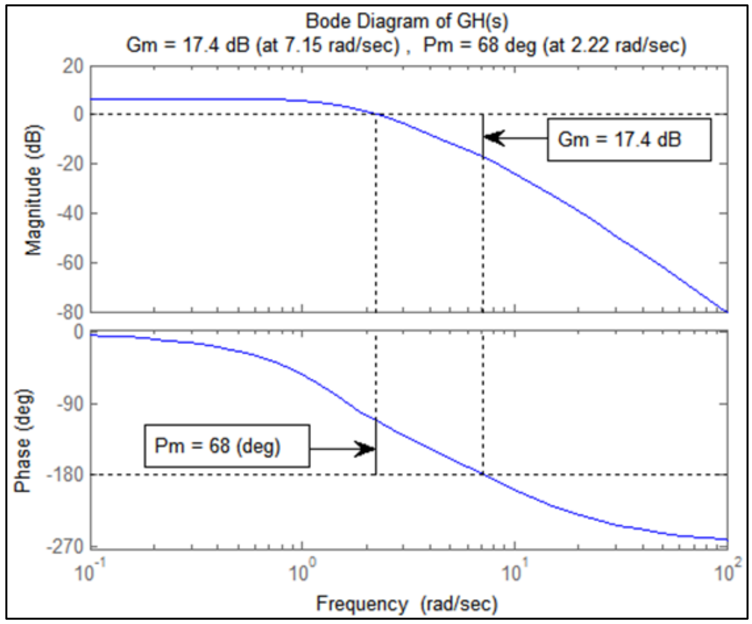
```
>> num = 100*[1,2];

>> den = conv ([1,5], conv([1,10],[1,2,2]));

>> sys = tf (num,den)

    Transfer function:
                 100 s + 200
    --------------------------------------------
    s^4 + 17 s^3 + 82 s^2 + 130 s + 100

>> margin(sys)
```

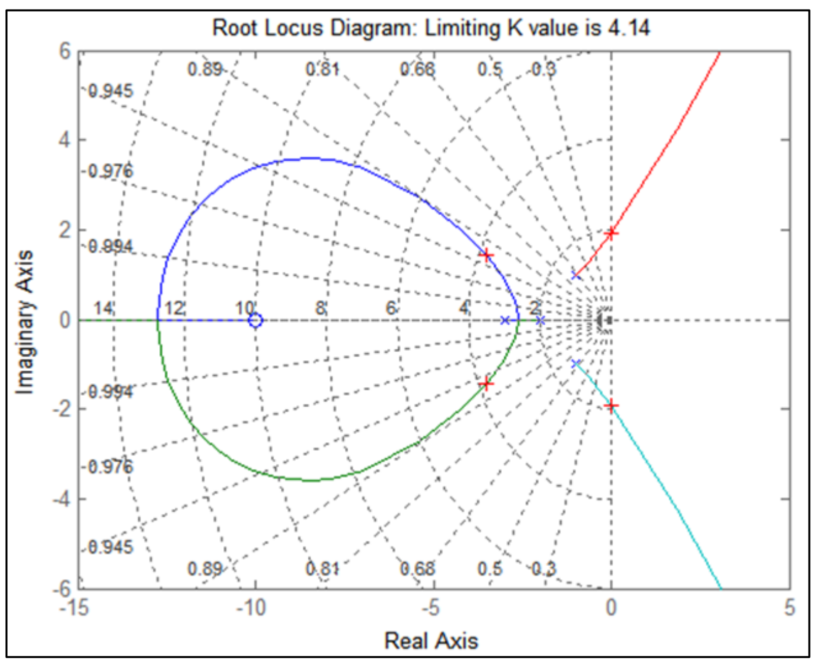


Comparison of Results from Root Locus and Bode Diagrams

Now we ***compare*** the results obtained from root locus and Bode diagrams for a single system. Consider the closed-loop system whose loop transfer function is

$$GH(s) = \frac{K\,(s+10)}{(s+2)(s+3)(s^2+2s+2)}$$

Here, the root locus diagram for the parameter K indicates the system is ***unstable*** when $K > 4.14$. Two poles lie and the imaginary axis.



Bode diagrams of the loop transfer functions for $K = 1$ and $K = 10$ are shown below. The diagrams show positive gain and phase margins for $K = 1$ and negative margins for $K = 10$.

The positive gain and phase margins indicate the system is ***stable*** for $K=1$ and the negative margins indicate the system is ***unstable*** for $K=10$. Using either Bode diagram, K_{max} the gain required to make the system ***marginally stable*** can be calculated. Using the first diagram, set

$$20\log_{10}(x)=12.3 \quad \Rightarrow \quad \boxed{x=10^{\left(12.3/20\right)}=4.12} \quad \Rightarrow \quad \boxed{K_{max}=K\cdot x=1\cdot x=4.12}$$

Using the second diagram, set

$$20\log_{10}(x)=-7.69 \quad \Rightarrow \quad \boxed{x=10^{\left(-7.69/20\right)}=0.413} \quad \Rightarrow \quad \boxed{K_{max}=K\cdot x=10\cdot x=4.13}$$