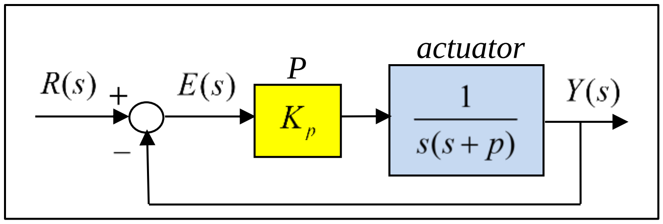
ME 4710 Motion and Control
PID Control of a Hydraulic Actuator: Root Locus Analysis

<u>Proportional (P) Control</u>

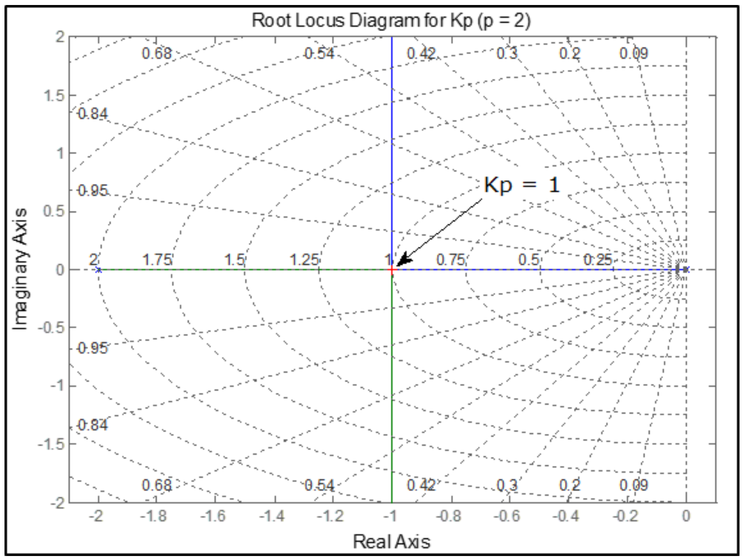
The diagram shows ***proportional control*** of a simple hydraulic actuator. The system has ***two parameters*** – K_p and p. K_p is the ***proportional gain***, and p is an ***actuator parameter*** that represents how quickly it responds to a command.



This is a ***type 1*** system. It has ***finite error*** for a ***ramp input***. If $K_p \geq p$, then the ***steady state ramp error*** is less than or equal to one. That is, $e_{ss} \leq 1$.

$$e_{ss} = \lim_{s \to 0}\left(s \cdot \frac{1}{s^2} \cdot \frac{E}{R} \right) = \lim_{s \to 0}\left(\cancel{s} \cdot \frac{1}{\cancel{s^2}} \cdot \frac{\cancel{s}(s+p)}{s^2+ps+K} \right) = \frac{p}{K} \qquad \text{(ramp input)}$$

Assuming the parameter $p = 2$, the root locus diagram for the proportional gain K_p is shown to the right. The system response can be divided into three categories depending on the value of K_p.



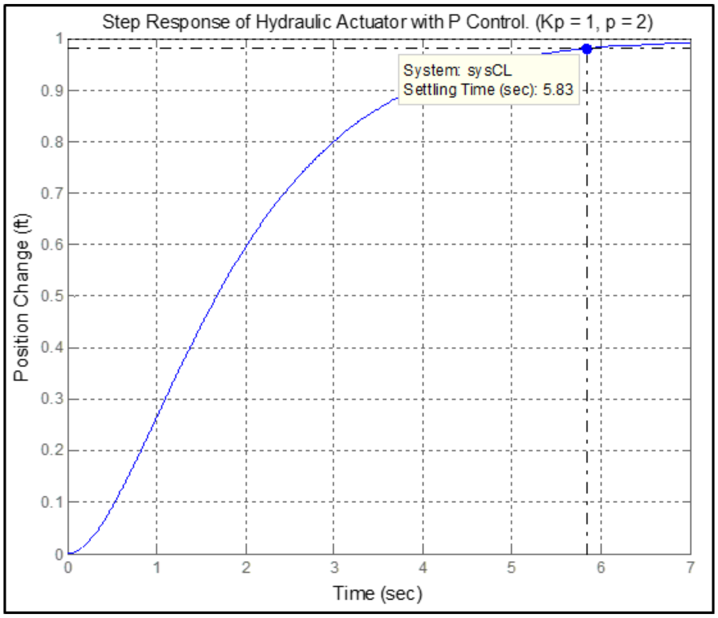
over-damped: $0 < K_p < 1$

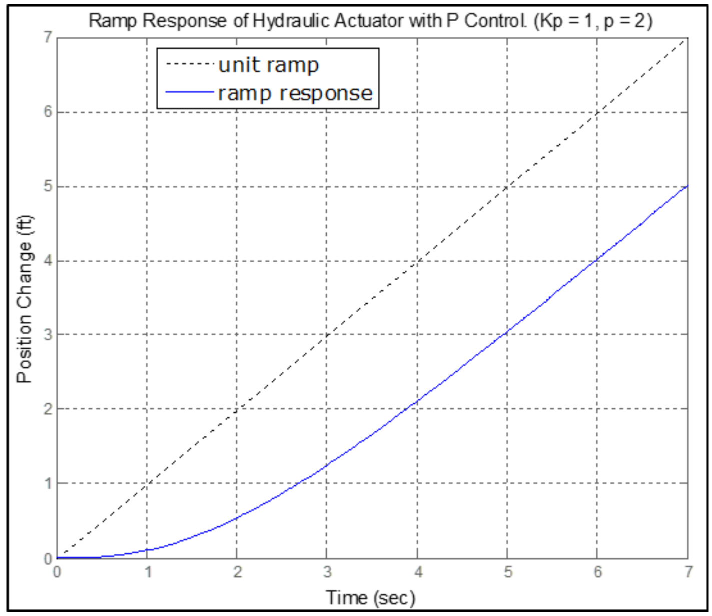
critically damped: $K_p = 1$

under-damped: $K_p > 1$

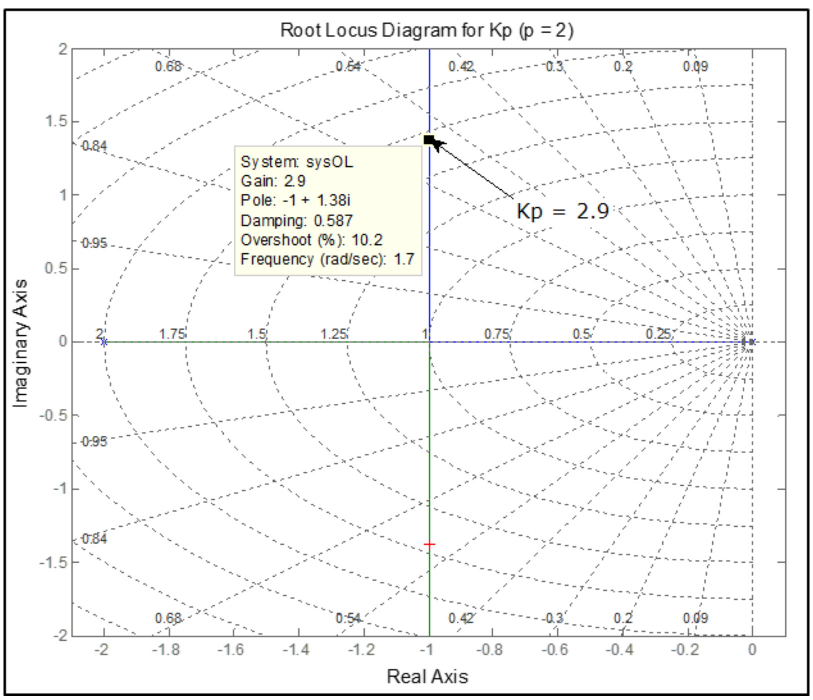
For $K_p \geq 1$ the real values of the poles are equal to -1 so the ***settling time*** of the system should be on the order of $T_s \approx 4/1 = 4$ (sec). For $K_p \gg 1$, the system response will become ***faster*** and more ***highly under-damped***, but the settling time remains unchanged.

The **step** and **ramp** responses of the system for $p = 2$ and $K_p = 1$ are shown in the diagrams below. Note that MATLAB® indicates the settling time is almost six seconds. Note also the steady-state ramp error is $e_{ss} = 2$ as calculated above.

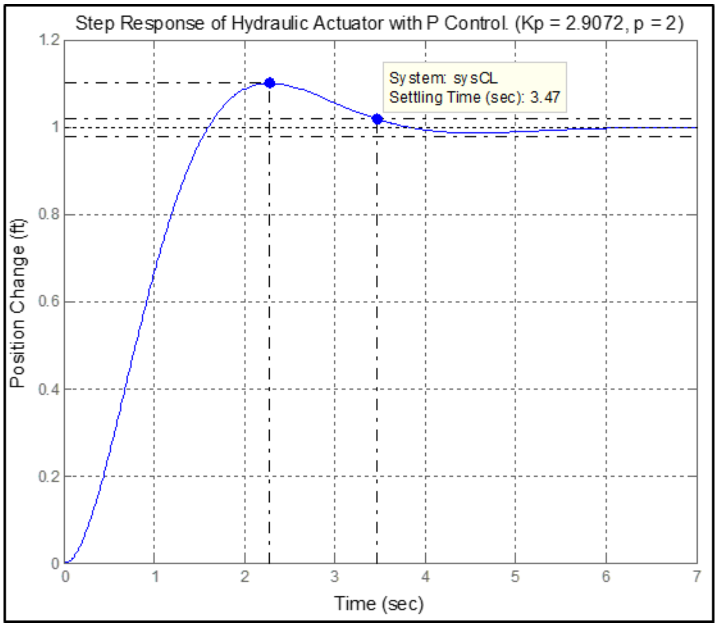


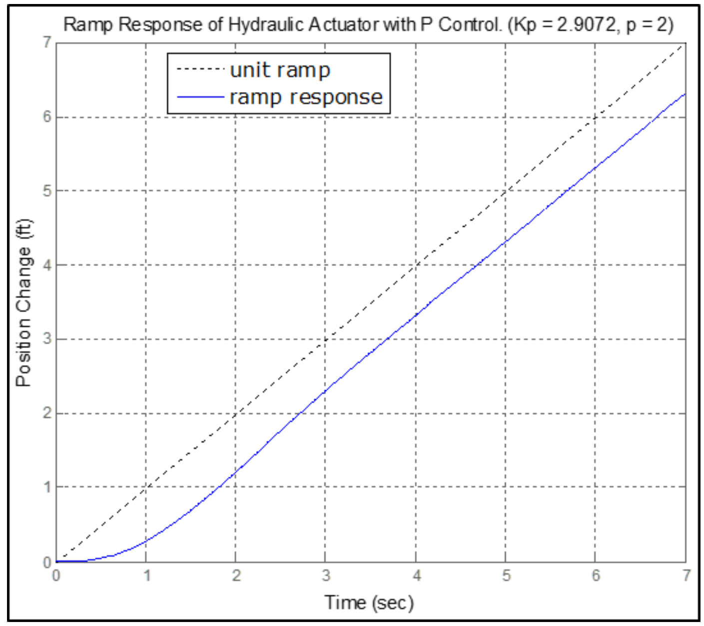


The root locus diagram below shows the locations of the closed-loop poles for a proportional gain $K_p \approx 2.9$. The pole locations are $-1 \pm 1.38i$.



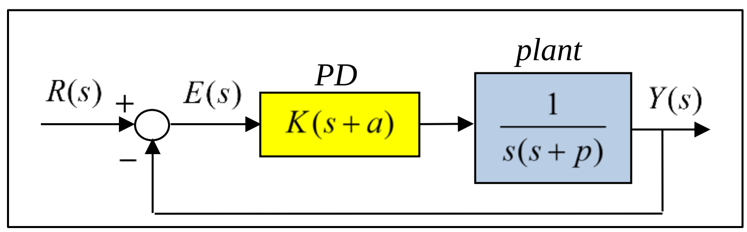
The step and ramp responses of the system for $p = 2$ and $K_p \approx 2.9$ are shown below. Note the settling time for the system is $T_s \approx 3.47$ (sec) (just under four seconds), and the ramp error is $e_{ss} = 2/2.9072 \approx 0.688$.





Proportional/Derivative (PD) Control

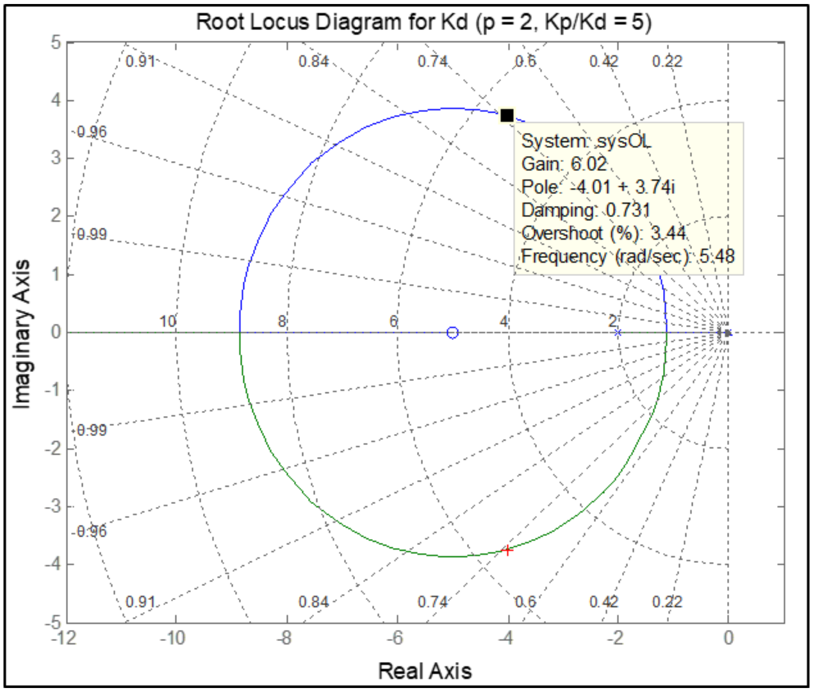
The actuator system is now shown with PD control. Here, $K = K_D$ is the derivative gain, and $a = K_P / K_D$ is the ***ratio*** of the proportional and derivative gains.



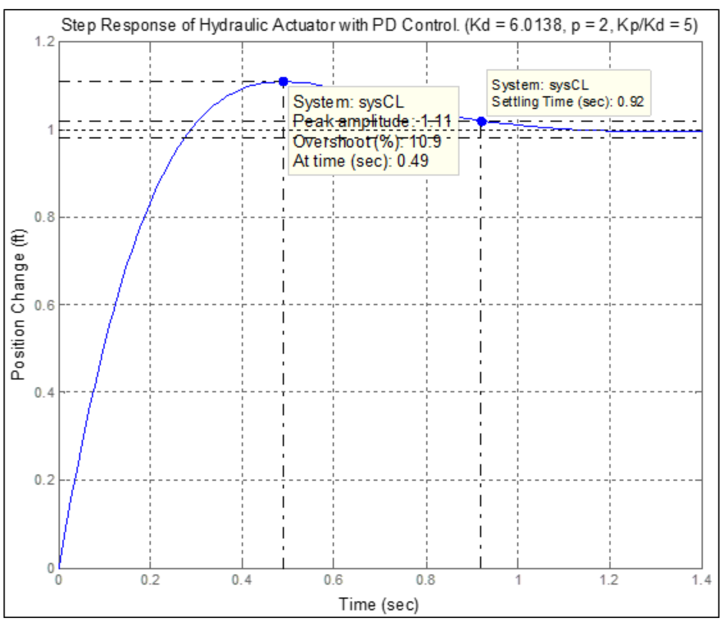
This is a ***type 1*** system, so it has a ***finite error*** for a ramp input. The steady state ramp error will be less than one $(e_{ss} \le 1)$ when $aK = K_P \ge p$.

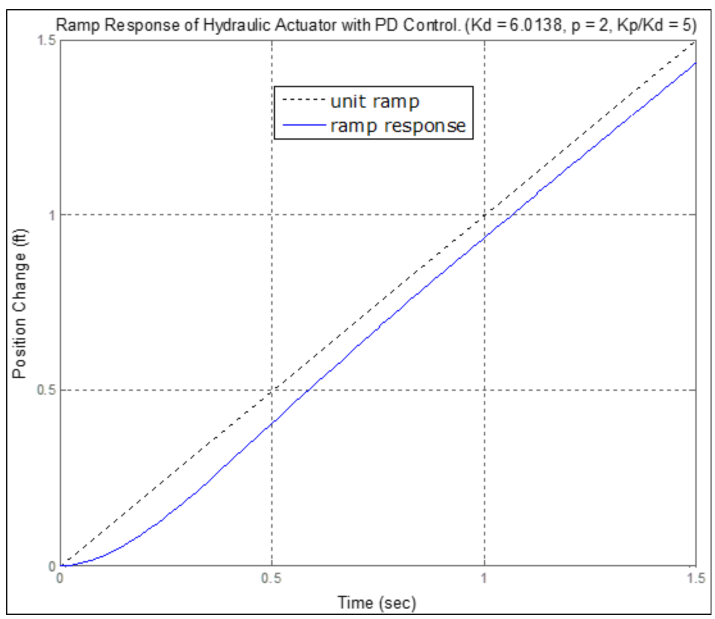
$$e_{ss} = \lim_{s\to 0}\left(s \cdot \frac{1}{s^2} \cdot \frac{E}{R} \right) = \lim_{s\to 0}\left(\cancel{s} \cdot \frac{1}{\cancel{s}^{\cancel{2}}} \cdot \frac{\cancel{s}(s+p)}{s^2 + (K+p)s + aK} \right) = \frac{p}{aK} = \frac{p}{K_P} \qquad \text{(ramp input)}$$

The root locus diagram for parameter K with $p = 2$ and $a = 5$ is shown below. The locations of the two poles for $K \approx 6$ are approximately $-4.01 \pm 3.74i$. As indicated in the diagram, the damping ratio and natural frequency for the poles are $\zeta \approx 0.73$ and $\omega_n \approx 5.48$ (rad/s), respectively. The settling time of the poles is $T_s \approx 4/4.01 \approx 1$ (sec).

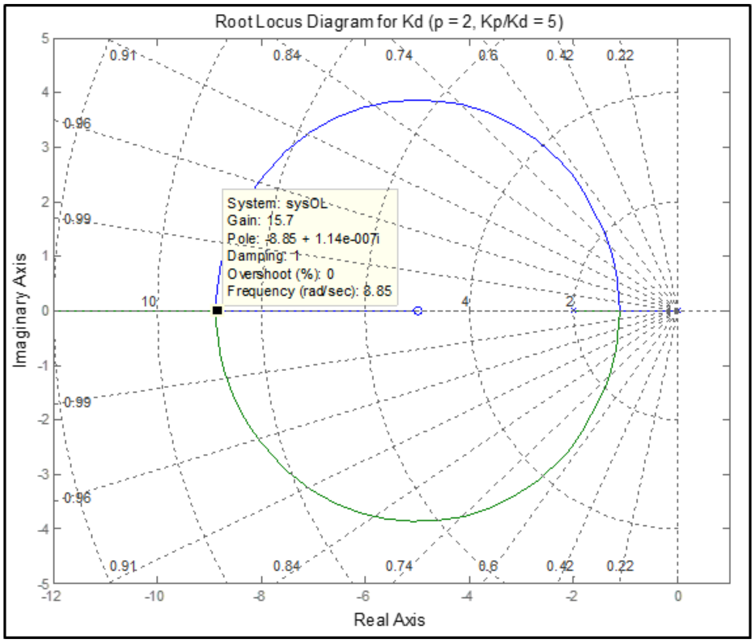


The step and ramp responses for $K \approx 6.0$, $p = 2$, and $a = 5$ are shown below. For these values, $K_P \approx 30.0$ and $e_{ss} \approx 0.07$. Note that the actual settling time is $T_s \approx 0.92 \approx 1$ (sec), and the percent overshoot for the step input is $\%OS \approx 11\%$. With no zero in the numerator, a second order system with $\zeta = 0.7$ has a little less than 5% overshoot. In this case, due to the presence of zero from the PD controller, the system overshoot is approximately double that value.

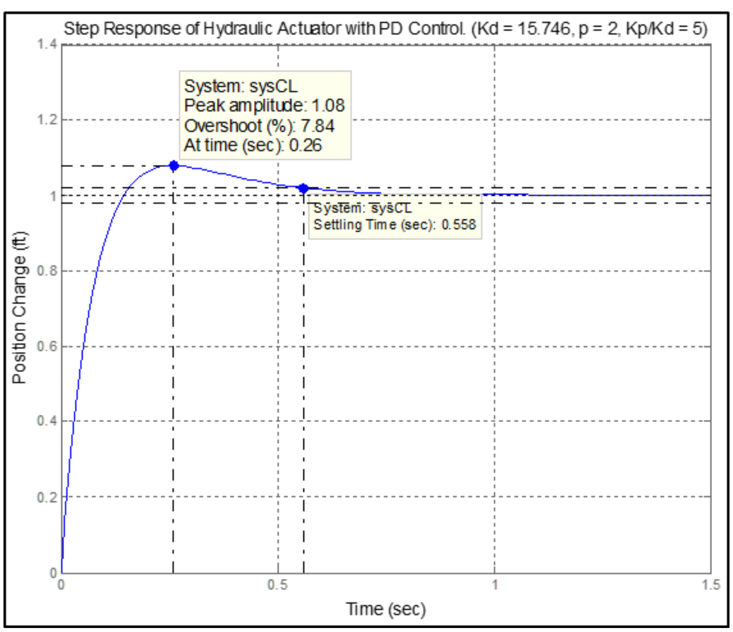


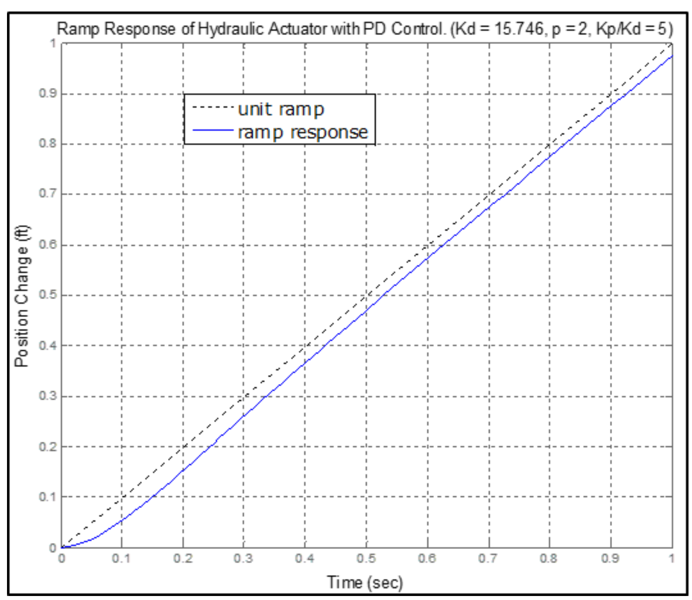


The root locus diagram for parameter K with $p = 2$ and $a = 5$ is shown again below. Here, the location of the poles for $K \approx 15.7$ is shown. In this case, the poles are very close to the real axis, indicating that the system is nearly critically damped $(\zeta \approx 1)$. The natural frequency of the poles is $\omega_n \approx 8.85$ (rad/s). The settling time of the poles is $T_s \approx 4/8.85 \approx 0.45$ (sec).



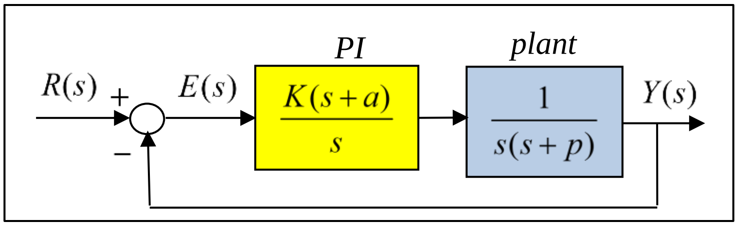
The step and ramp responses for $K \approx 15.7$, $p = 2$, and $a = 5$ are shown below. For these values, $K_P \approx 78.5$ and $e_{ss} \approx 0.0255$. Note the settling time is $T_s \approx 0.558$ (sec), and the percent overshoot is $\%OS \approx 7.84\%$. Ideally, a critically damped second-order system should have no overshoot, but the presence of the zero from the PD controller causes some overshoot and has increased the settling time somewhat.



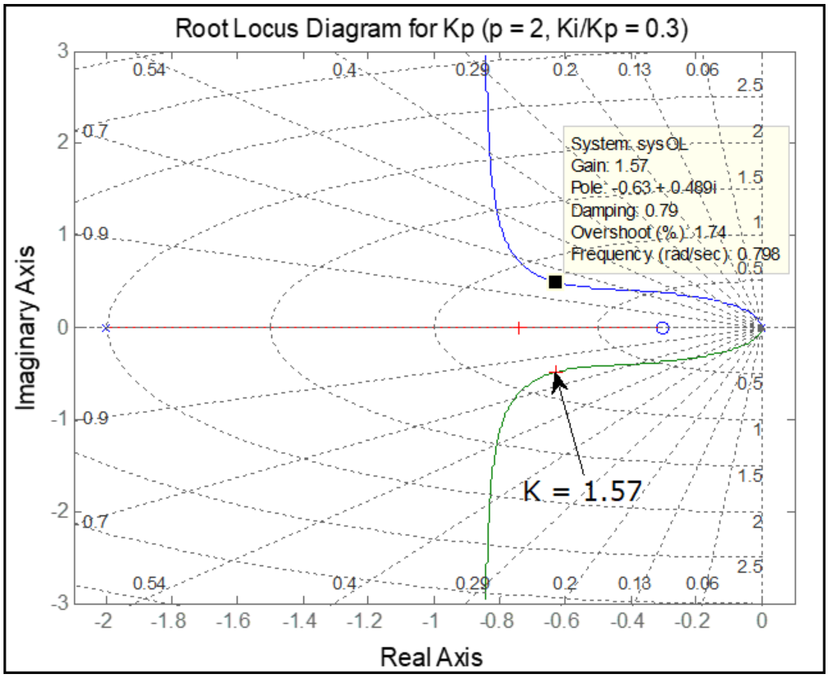


Proportional/Integral (PI) Control

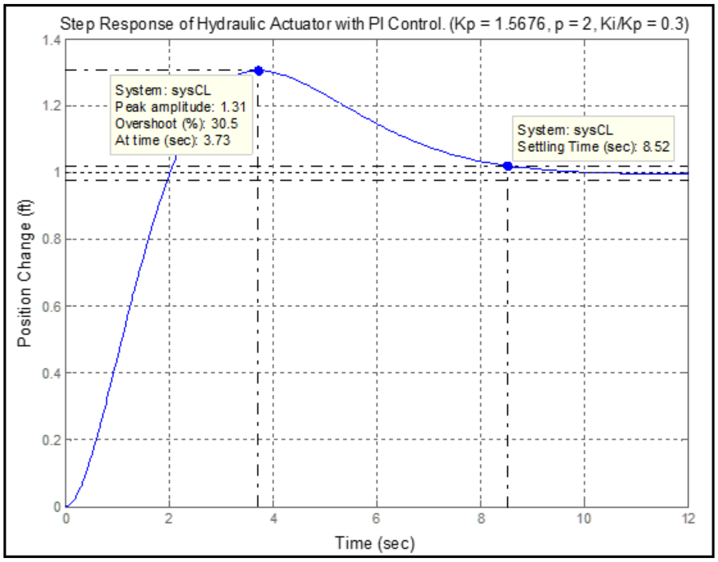
The actuator system is now shown here with PI control. Here, $K = K_P$ is the proportional gain, and $a = K_I / K_P$ is the ratio of the integral and proportional gains.

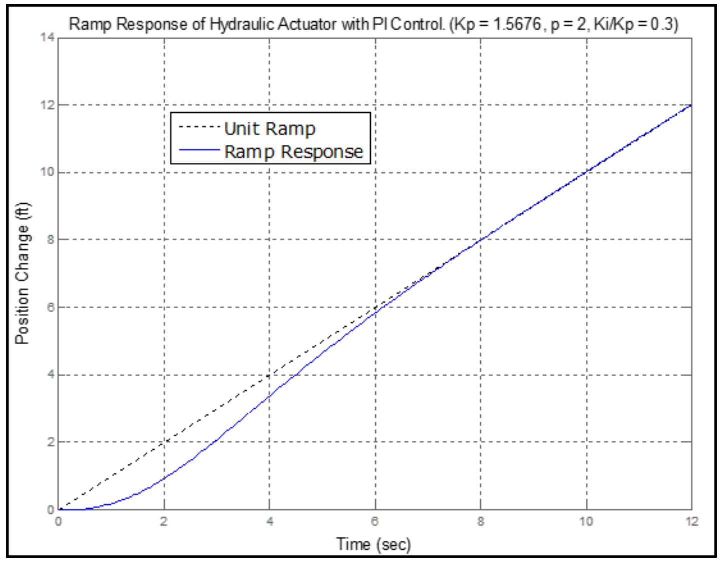


This is a ***type 2*** system and has ***zero steady-state error*** to ***both*** step and ramp inputs. The root locus diagram for the parameter K with $p = 2$ and $a = 0.3$ is shown below. The three poles for $K \approx 1.57$ are indicated on the diagram. The three roots are clustered together, so none of the roots dominate the response. The pair of complex conjugate poles are the slowest with a settling time $T_s \approx 4/0.63 \approx 6.35$ (sec).

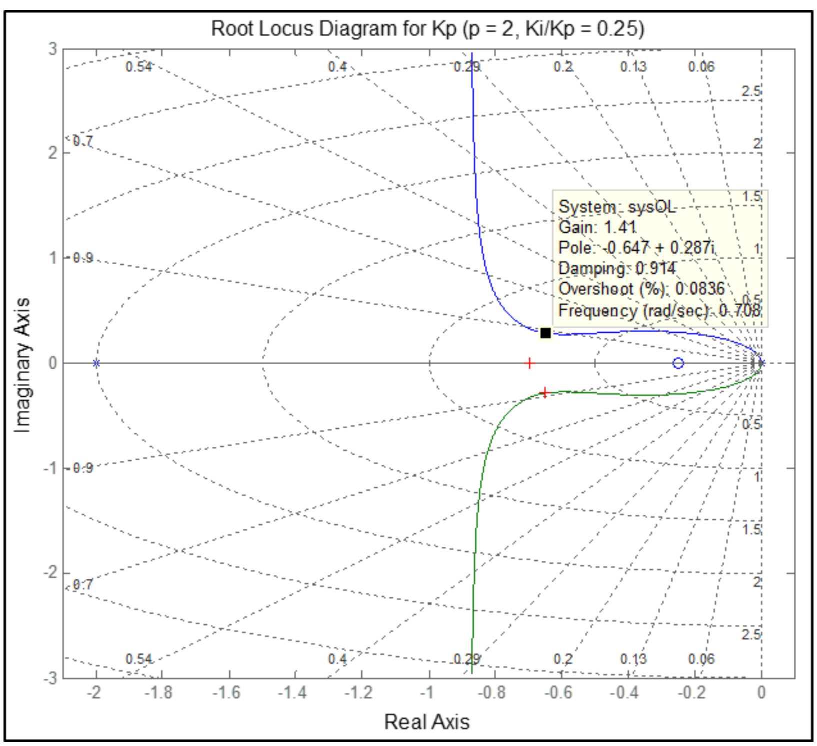


The step and ramp responses for $K \approx 1.57$ are shown below. The presence of the controller's zero at $s = -0.3$ (which is well inside the poles) amplifies the percent overshoot and significantly increases the settling time of the step response. The complex poles would indicate a percent overshoot of 1.74% and a settling time of 6.35 (sec), but the system has an overshoot of 30.5% and a settling time of 8.52 (sec). This over-responsiveness in the step response has favorable consequences in the ramp response. By 6.35 (sec), the system is very nearly reproducing the unit ramp input. The step response is not so good, but the ramp response is very good.

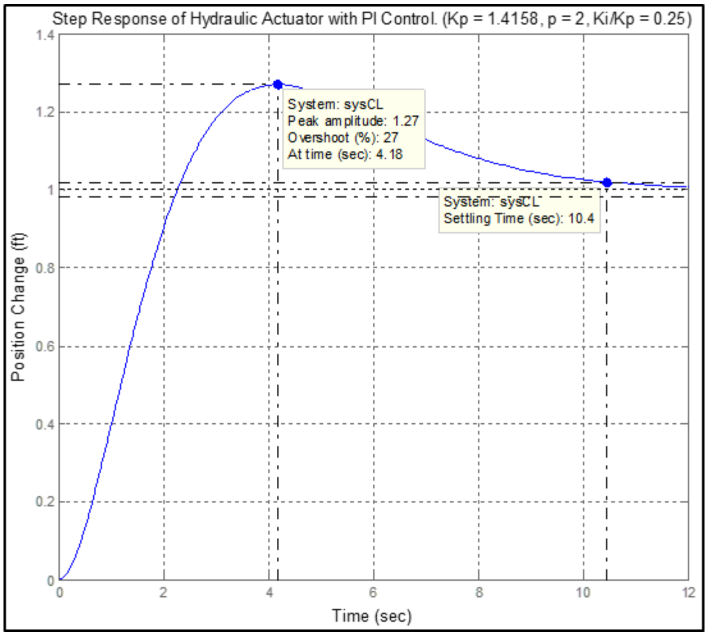


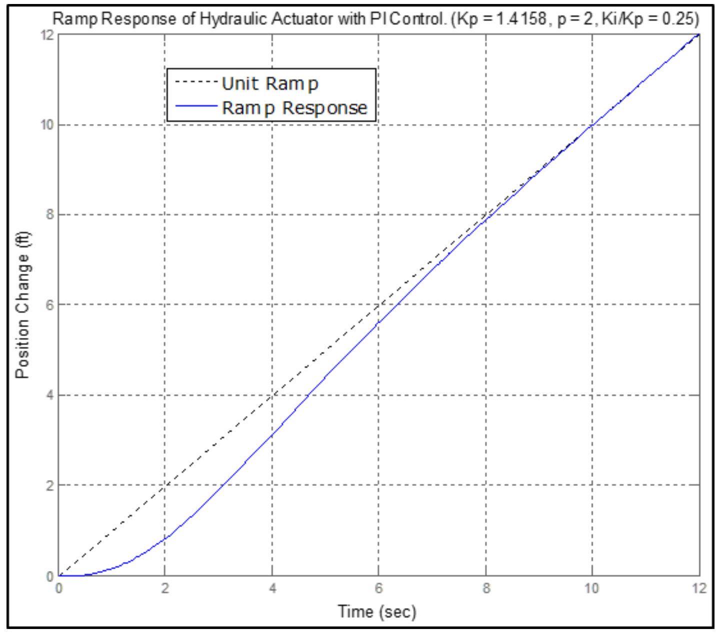


Now, consider moving the controller's zero closer to the imaginary axis. The root locus diagram for the parameter K with $p=2$ and $a=0.25$ is shown below. Note that the complex branches have been pulled closer to the real axis allowing for the choice of complex poles with a higher damping ratio. The three system poles for $K \approx 1.41$ are indicated on the diagram. As before, the three poles are located close together. The slower complex poles have a damping ratio greater than 0.9 and a settling time about 6.2 (sec). Unfortunately, the controller's zero is still well inside the three poles.



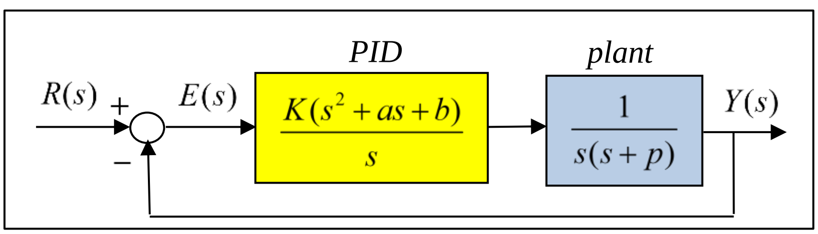
The step and ramp responses for $p = 2$, $a = 0.25$, and $K \approx 1.41$ are shown below. The ***overshoot*** of the ***step response*** is ***lowered*** (from the previous PI controller), but the ***settling times*** for the step and ramp responses are both ***higher***. In this case, little benefit has been achieved by increasing the damping ratio of the complex poles.





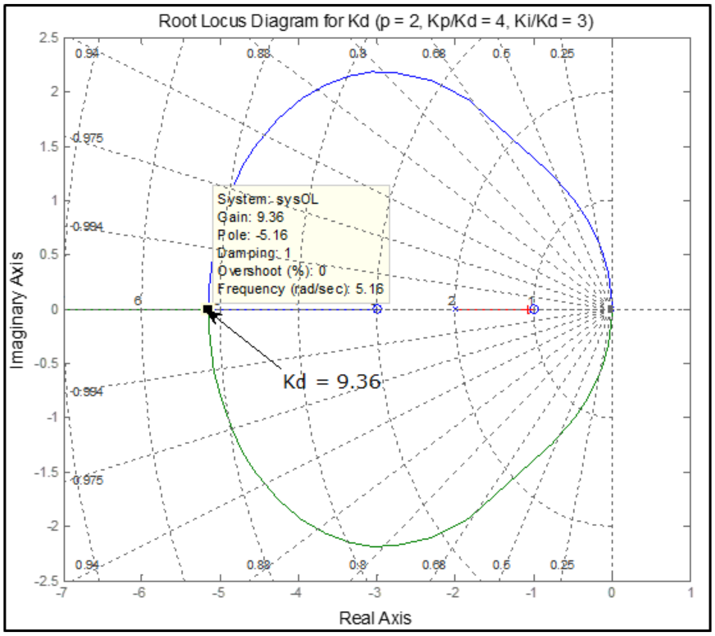
Proportional/Integral/Derivative (PID) Control

The actuator system is shown here with PID control. Here, $K = K_D$ is the derivative gain, $a = K_P / K_D$ is the ratio of the proportional and derivative gains, and $b = K_I / K_D$ is the ratio of the integral and derivative gains.

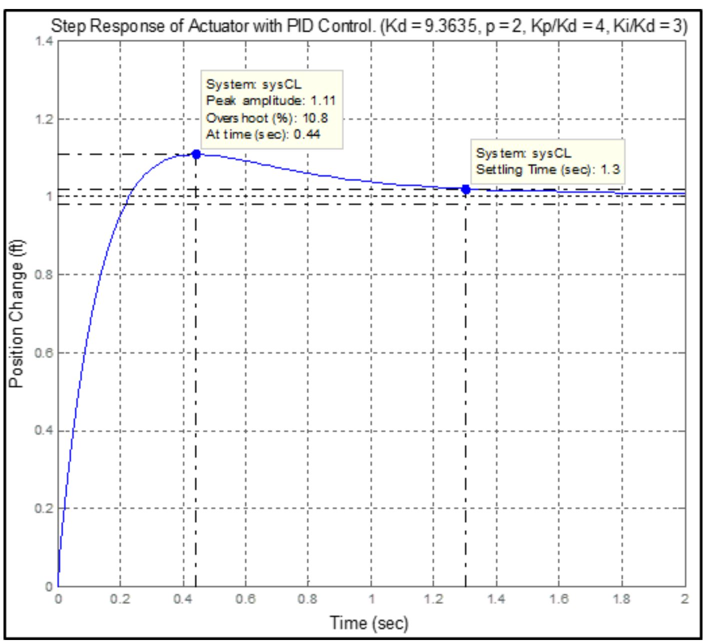


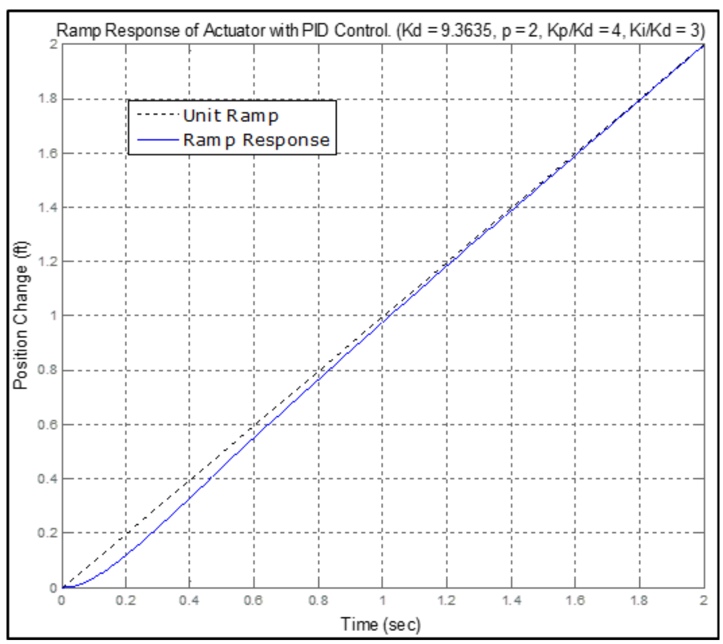
This is a ***type 2*** system and has ***zero steady-state error*** to ***both*** step and ramp inputs.

The root locus diagram for the parameter $K = K_D$ with $p = 2$, $a = K_P / K_D = 4$, and $b = K_I / K_D = 3$ is shown below. The three pole locations for $K = K_D \approx 9.36$ are indicated on the diagram. One of the poles nearly cancels the controller zero at $s = -1$, and the other poles are two real, repeated poles at $s = -5.16$. For this value of *K*, the system has two real, equal poles with a zero inside of the poles. The presence of the zero is expected to cause some overshoot and to extend the settling time associated with the repeated poles.



The step and ramp responses for $p = 2$, $a = K_P / K_D = 4$, $b = K_I / K_D = 3$, and $K = K_D \approx 9.36$ are shown below. The step response shows a 10.8% overshoot and a settling time of 1.3 (sec). Note that without the presence of the zero, the system would have no overshoot and a settling time of $T_s \approx 4/9.36 \approx 0.43$ (sec). The ramp response acquires the unit ramp in under 2 (sec).





Comparison of Responses:

Clearly the PID controlled response is the ***best*** response. It is somewhat slower than the PD controlled response, but it has zero steady state error for both step and ramp responses.