

ME 4710 Motion and Control

PID Control of a Spring-Mass-Damper Position (Root Locus Analysis)

Fig. 1 shows a spring-mass-damper (SMD) system with a *force actuator* for *position control*. The spring has stiffness k , the damper has coefficient c , the block has mass m , and the position of the mass is measured by the variable x . The transfer function of the SMD with an actuating force F_a as input and the position x as output

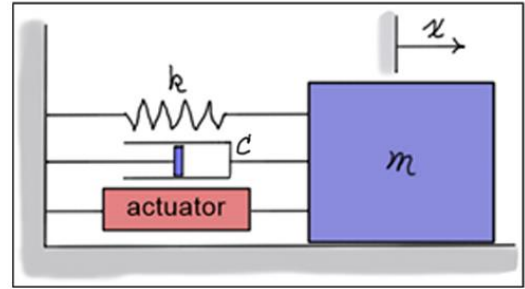
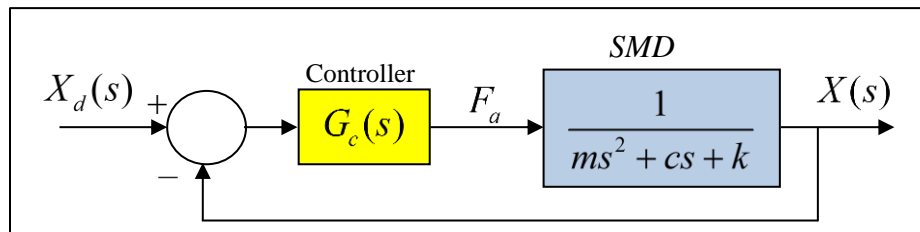


Fig. 1. Spring-Mass-Damper System with Force Actuator

is

$$\boxed{\frac{X}{F_a}(s) = \frac{1}{ms^2 + cs + k}} \quad (1)$$

Assuming *ideal* actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, X_d represents the *desired position*, and $G_c(s)$ represents the *transfer function* of the controller.



For the analyses that follow, it is assumed the SMD *parameters* are: $m = 1$ slug, $c = 8.8$ (lb-s/ft), and $k = 40$ (lb/ft). This represents an *under-damped, second-order* plant with

$$\text{Natural Frequency: } \boxed{\omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)}}$$

$$\text{Damping Ratio: } \boxed{\zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7}$$

Proportional Control

For *proportional control*, $\boxed{G_c(s) = K}$, and the loop and closed-loop transfer functions are

$$\boxed{GH(s) = \frac{K}{s^2 + 8.8s + 40}} \quad \boxed{\frac{X}{X_d}(s) = \frac{K}{s^2 + 8.8s + (40 + K)}} \quad (2)$$

Using $GH(s)$, the RL diagram for the closed-loop system for $K \geq 0$ is shown in **Fig. 2**. Note that as the value of K is *increased*, the complex, closed-loop poles move straight up/down, indicating that the *natural frequency* is *increased* and the *damping ratio* is *decreased* as K is increased.

This is a *type-zero* system and hence will have a *finite steady-state error* for a step input.

Using the final-value theorem and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$x_{ss} = \lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot \frac{K}{s^2 + 8.8s + (40 + K)} \right) = \frac{K}{40 + K} < 1 \quad (3)$$

Eq. (3) indicates that *large values* of K *lead to small steady-state errors*; however, they also lead to *faster, less damped responses*.

This conclusion is verified in **Fig. 3** which shows the closed-loop step responses for proportional control gains K of 100, 500, and 2000. Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. To remove the steady-state error and have better response, integral and/or derivative terms must be included.

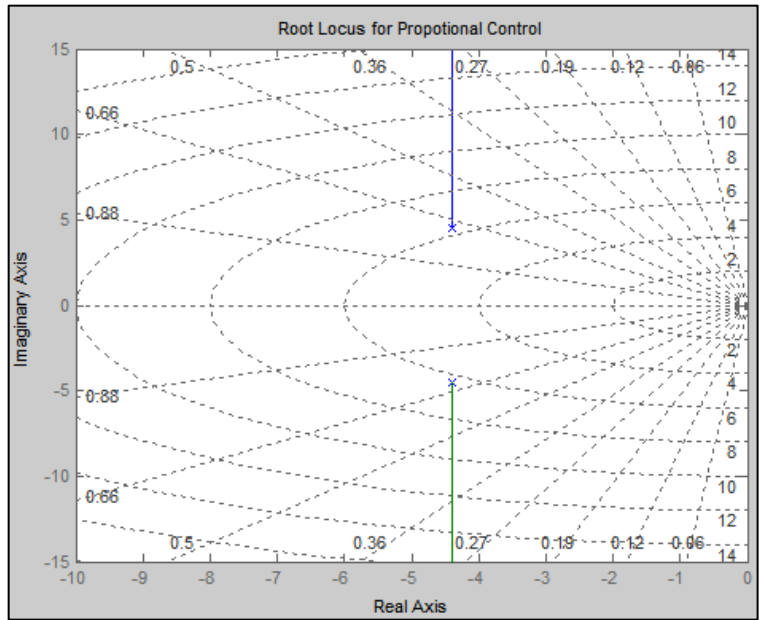


Fig. 2. RL Diagram for Proportional Control

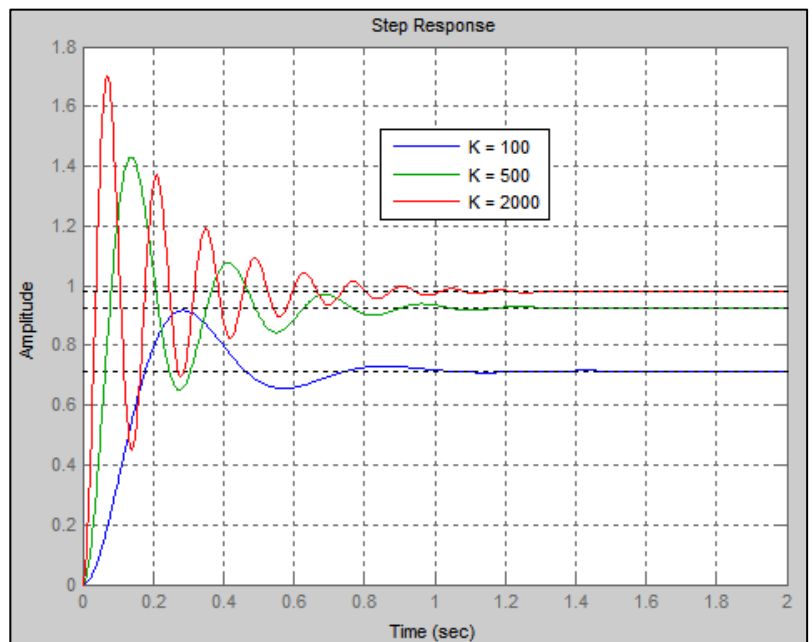


Fig. 3. Step Response for Various Proportional Control Gains

Proportional-Integral (PI) Control

For *proportional-integral (PI) control*

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p(s+a)}{s} \quad (4)$$

Here, the coefficients K_p and K_I represent the *proportional* and *integral gains*, and the coefficient $a = K_I/K_p$ is the *ratio* of the integral and proportional gains. In this case, the loop and closed-loop transfer functions are

$$GH(s) = \frac{K_p(s+a)}{s(s^2 + 8.8s + 40)} \quad \frac{X}{X_d}(s) = \frac{K_p(s+a)}{s(s^2 + 8.8s + 40) + K_p(s+a)} \quad (5)$$

Using *integral control* makes the system *type-one*, so the *steady-state error* due to a step input is *zero*. This can be verified using the final value theorem to show that $x_{ss} = 1$ when the input is a unit step function. **Fig. 4** shows the RL diagram for the closed-loop system with $a = 3$. It also shows the location of the closed-loop poles for a proportional gain $K_p \approx 50$. **Fig. 5** shows the *closed-loop step response* for $a = 3$ and $K_p = 25, 50$, and 75 .

Integral control has *removed* the *steady-state error* and *improved* the *transient response*, but it has also *increased* the system's *settling time*.

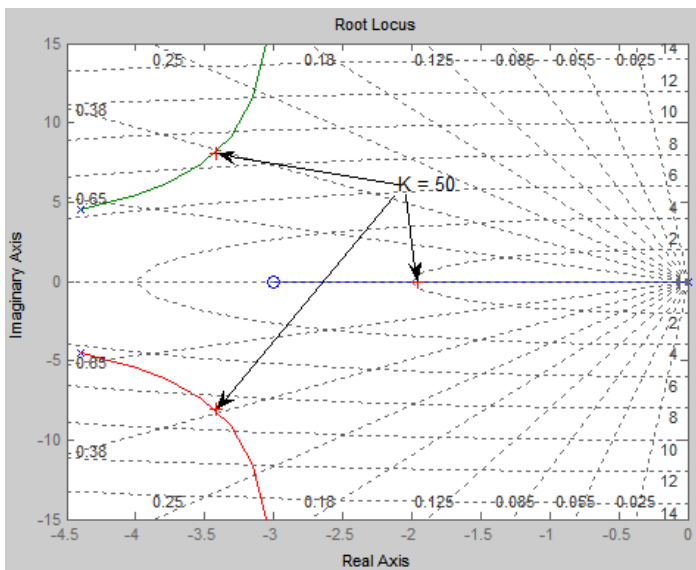


Fig. 4. Root Locus Diagram for PI Control ($a = 3$)

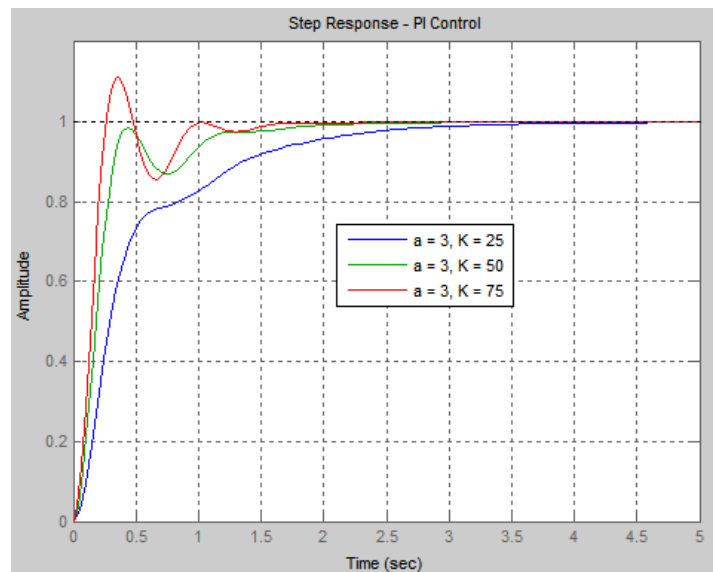


Fig. 5. Step Response for PI Control ($a = 3$) for Various Proportional Gains

Proportional-Derivative (PD) Control

For **proportional-derivative (PD) control**:
$$G_c(s) = K_p + K_D s = K_D(s + a) \quad (6)$$

The coefficients K_p and K_D represent the **proportional** and **derivative gains**, and $a = K_p/K_D$ is the ratio of the proportional and derivative gains. The loop and closed-loop transfer functions are

$$GH(s) = \frac{K_D(s + a)}{s^2 + 8.8s + 40} \quad \frac{X}{X_d}(s) = \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \quad (7)$$

Without integral control, this is a **type-zero** system, and hence will have a **finite steady-state error** to a unit step input. Using the final-value theorem and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$x_{ss} = \lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \right) = \frac{K_D a}{40 + K_D a} = \frac{K_p}{40 + K_p} < 1 \quad (8)$$

As with proportional control, **larger proportional gains** produce **smaller steady-state errors**. **Fig. 6** shows the RL diagram for the closed-loop system with $a = 10$. It also shows the location of the closed-loop poles for $K_D \approx 25.6$. **Fig. 7** shows the **closed-loop step response** for $a = 10$ and derivative gains of $K_D = 10, 27, 50,$ and 75 . The PD controller has **decreased** the system's **settling time** considerably; however, to control the steady-state error, the derivative gain K_D must be high. This **decreases** the **response times** of the system and can make it **susceptible to noise**.

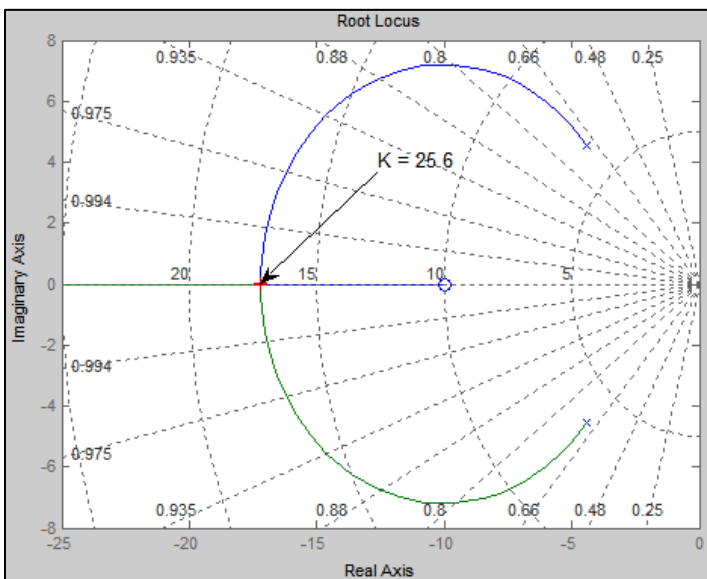


Fig. 6. Root Locus Diagram for PD Control ($a = 10$)

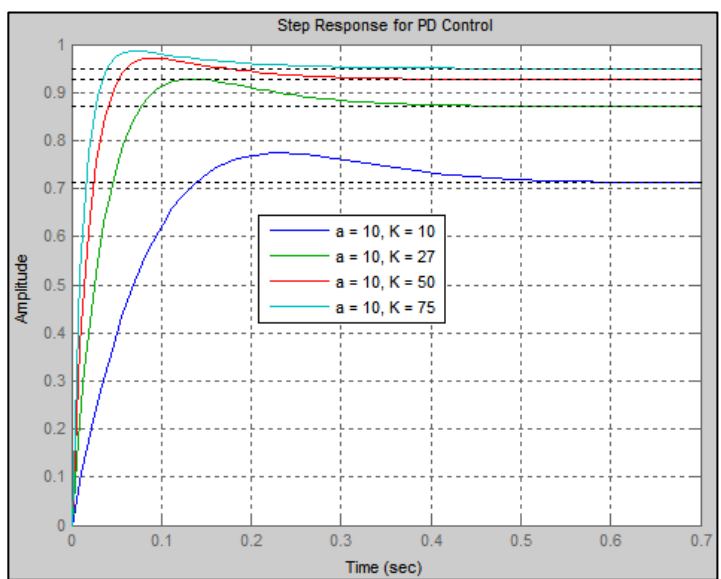


Figure 7. Step Response for PD Control ($a = 10$) for Various Derivative Gains

Proportional-Integral-Derivative Control

For *proportional-integral-derivative (PID) control*

$$G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s} \quad (9)$$

The coefficients K_p , K_I , and K_D represent the *proportional*, *integral*, and *derivative gains*, $a = K_p/K_D$ is the *ratio* of the proportional and derivative gains, and $b = K_I/K_D$ is the *ratio* of the integral and derivative gains. In this case, the loop and closed-loop transfer functions are

$$GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)} \quad \frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)} \quad (10)$$

Again, with *integral control*, the system is *type-one* and has *zero steady-state error* for a step input. **Fig. 8** shows the RL diagram of the closed-loop system for $a = 15$ and $b = 50$. The location of the closed-loop poles for $K_D \approx 15.8$ is also shown. **Fig. 9** shows the step response of the closed-loop system for $a = 15$, $b = 50$, and various derivative gains.

The PID controller has *removed steady-state error* and *decreased* the system's *settling times* while maintaining a *reasonable transient response*.

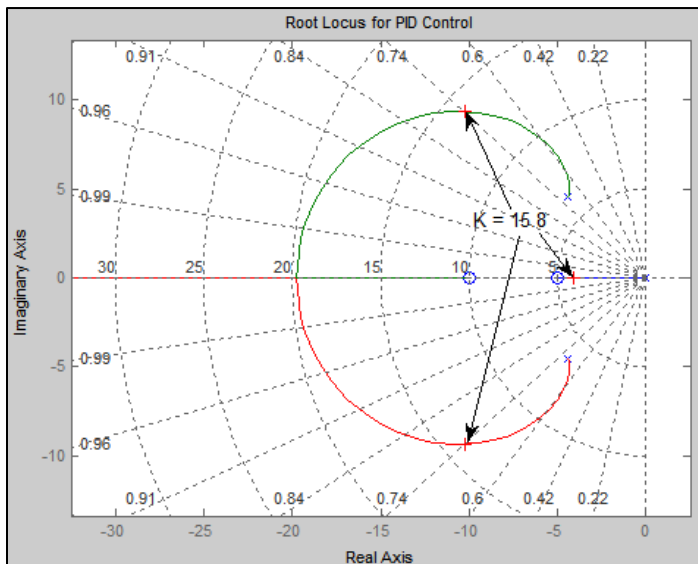


Fig. 8. Root Locus Diagram for PID Control
($a = 15, b = 50$)

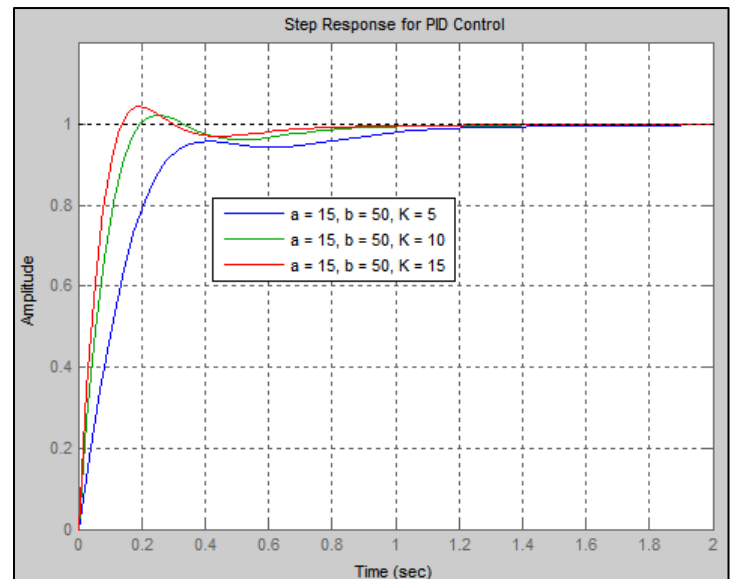


Fig. 9. Step Response for PID Control ($a = 15$)
($b = 50$) for Various Derivative Gains