ME 4710 Motion and Control
PID Control of a Spring-Mass-Damper Position: Analysis Summary

Fig. 1 shows a spring-mass-damper system with a force actuator for position control. The spring has stiffness k, the damper has coefficient c, the block has mass m, and the position of the mass is measured by the variable x. The *transfer function* of the SMD with the actuating force F_a as input and the position x as output is
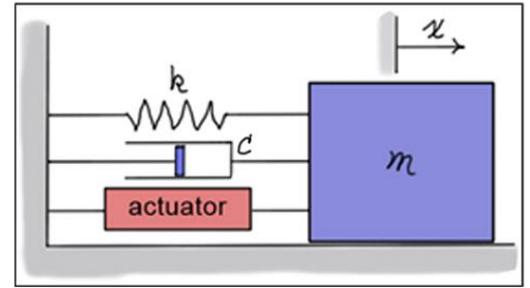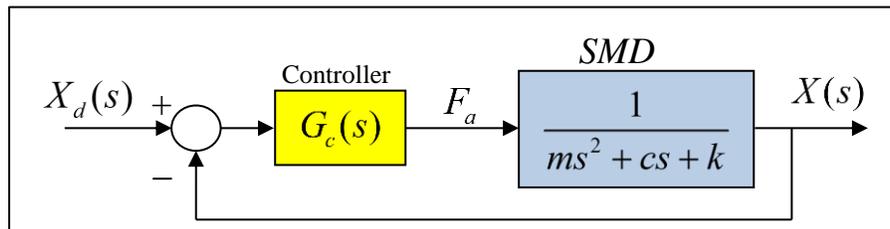

Fig. 1. Spring-Mass-Damper System with Force Actuator

$$\boxed{\frac{X}{F_a}(s) = \frac{1}{ms^2 + cs + k}} \qquad (1)$$

Assuming ideal actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, X_d represents the *desired position*, and $G_c(s)$ represents the *transfer function* of the controller.

It is assumed here that the SMD parameters are: $m = 1$ slug, $c = 8.8$ (lb-s/ft), and $k = 40$ (lb/ft). This represents an under-damped, second-order plant with the following *natural frequency* and *damping ratio*.

$$\boxed{\begin{array}{l} \omega_n = \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)} \\ \zeta = \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7 \end{array}}$$

Proportional Control

If *proportional control* is used, then $\boxed{G_c(s) = K}$, and the *loop transfer function* and *closed-loop transfer functions* are

$$\boxed{GH(s) = \frac{K}{s^2 + 8.8s + 40}} \qquad \boxed{\frac{X}{X_d}(s) = \frac{K}{s^2 + 8.8s + (40 + K)}} \qquad (2)$$

This is a ***type-zero*** system and will have a ***finite steady-state error*** for a step input. Using the final-value theorem and the closed-loop transfer function, x_{ss} the ***final value*** of $x(t)$ to a ***unit step*** command is

$$x_{ss} = \lim_{s \to 0} \left(\cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{K}{s^2 + 8.8s + (40 + K)} \right) = \frac{K}{40 + K} < 1 \qquad (3)$$

Eq. (3) indicates that ***large values*** of K ***lead to smaller steady-state errors***; however, as seen below, they also lead to a ***faster, less damped responses***.

The root locus diagram for the closed-loop system for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in ***Fig. 2***. Note that as the value of K is ***increased***, the closed-loop poles move straight up/down, indicating the natural frequency is ***increased*** and the damping ratio is ***decreased***. Also, as the value of K is ***increased***, the ***phase (stability) margin is decreased.***
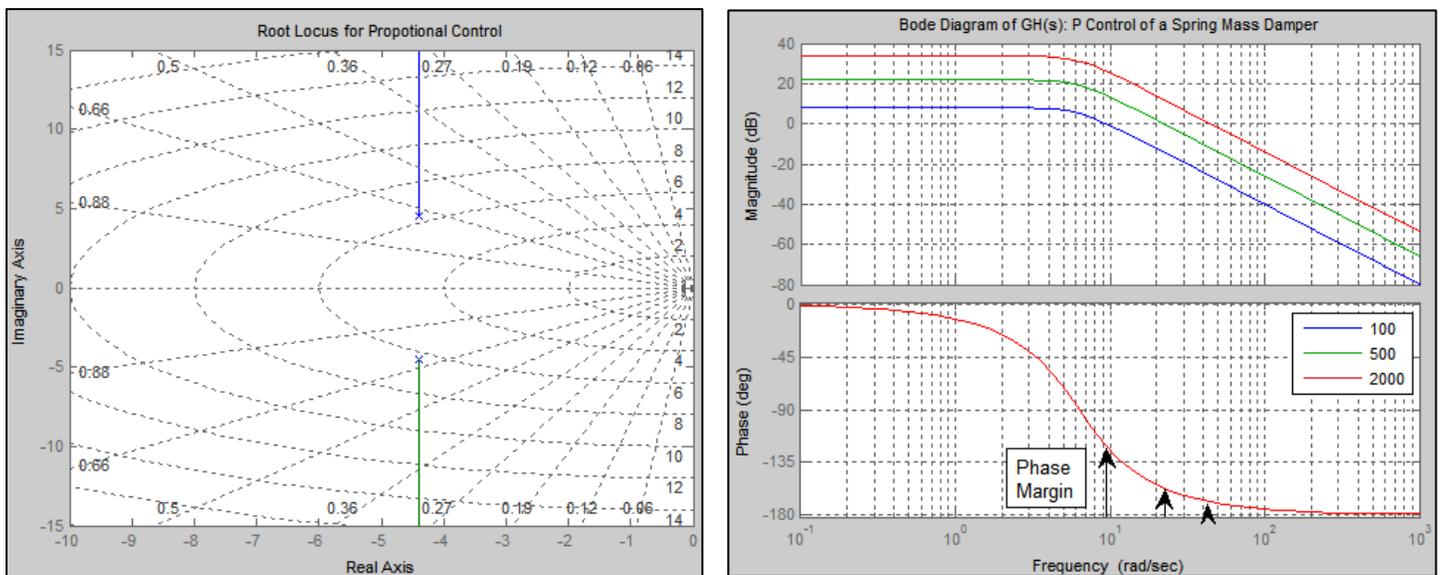
Fig. 2. Root Locus Diagram and Bode Diagram for ($GH(s)$) for Proportional Control

Fig. 3 shows step responses and Bode diagrams of the closed-loop system for proportional gains K of 100, 500, and 2000. As the gain is ***increased*** the system response is ***faster*** and ***less damped***. The Bode diagram correspondingly shows ***larger bandwidths*** and ***larger resonant magnitudes***. Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. As the gain is increased, the response becomes ***faster***, but it has a ***lower phase margin***. To remove the steady-state error and have better response, integral and/or derivative terms must be included in the controller.
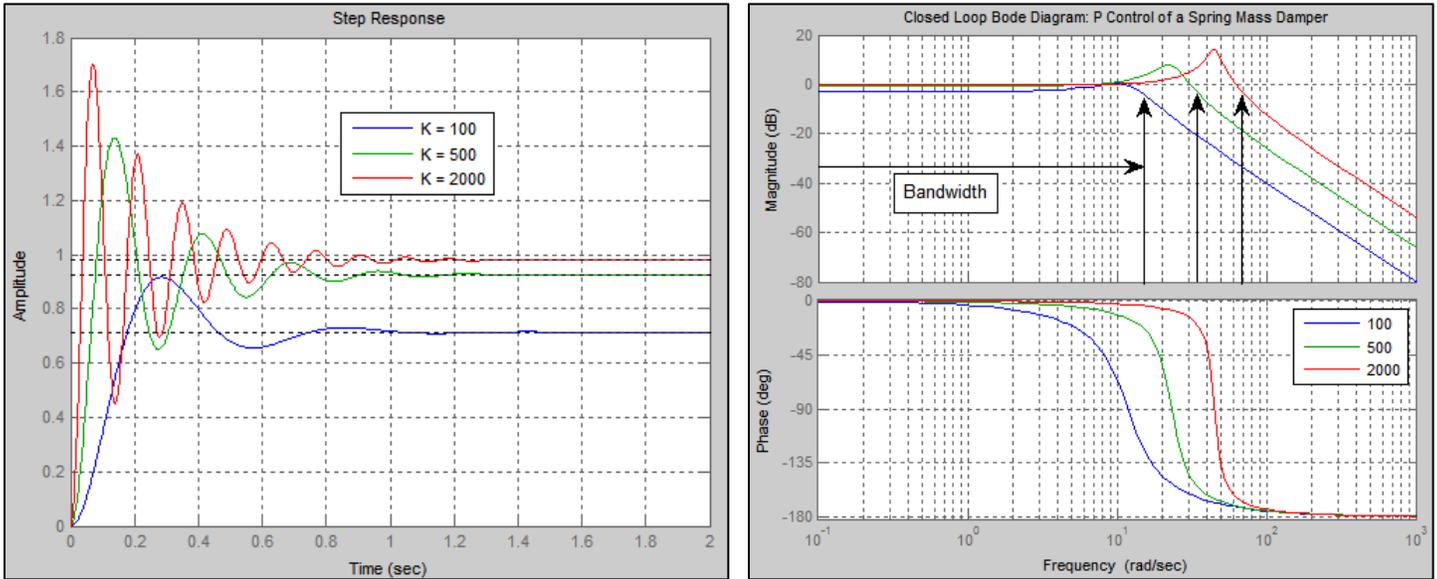
Fig. 3. Closed Loop Step Response and Bode Diagrams for P Control

Proportional-Integral (PI) Control

If **proportional-integral (PI) control** is used, then

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p(s+a)}{s} \tag{4}$$

The parameters K_P and K_I represent the **proportional** and **integral** gains, and the parameter $a = K_I/K_P$ is the **ratio** of the integral and proportional gains. The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_P(s+a)}{s(s^2 + 8.8s + 40)} \qquad \frac{X}{X_d}(s) = \frac{K_P(s+a)}{s(s^2 + 8.8s + 40) + K_P(s+a)} \tag{5}$$

Integral control makes the system a **type-one** system, so the **steady-state error** due to a step input is **zero**. This can be verified using the final value theorem to show that $x_{ss} = 1$ for a unit step input.

The root locus diagram for the closed-loop system (with $a = 3$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in **Fig. 4**. The root locus diagram also shows the locations of the closed-loop poles for a proportional gain $K_P = 50$. Note the integral controller has **added a third, slower pole** to the system and has **moved the asymptotes** of the complex poles closer to the imaginary axis. For low gains, the system is **slow and stable** (first order dominant).

As the proportional gain is ***increased***, the system becomes ***faster*** with a ***decreasing phase margin***. The Bode diagram shows that the gain could be increased somewhat above $K_p = 25$ without significantly decreasing the stability margin. However, further increases will decrease the phase margin.
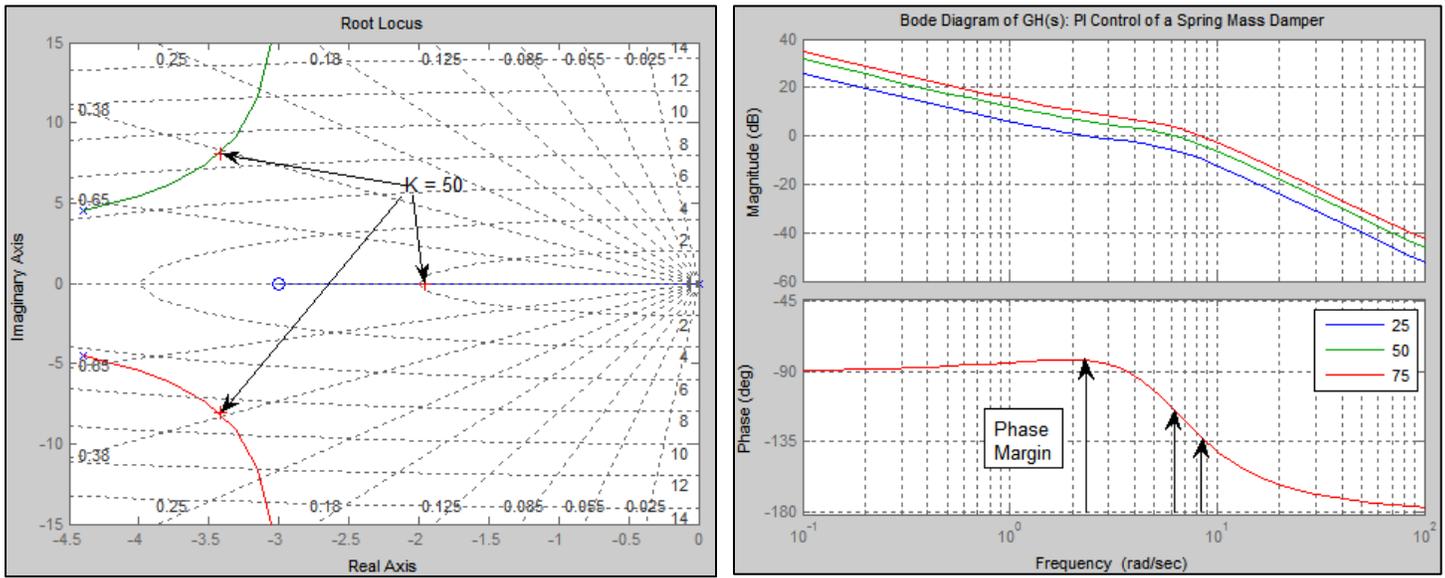

Fig. 4. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PI Control ($a = 3$)

Fig. 5 shows step responses and Bode diagrams of the closed loop system for $a = 3$ and proportional gains of $K_p = 25, 50$, and 75. Integral control has ***removed*** the ***steady-state step error*** and ***improved*** the ***transient response***, but it has also ***increased*** the ***system's settling time***. Settling times can be lowered by ***increasing*** the gain. This will ***increase*** the ***system bandwidth***, but it will also ***decrease*** the ***stability margin***.
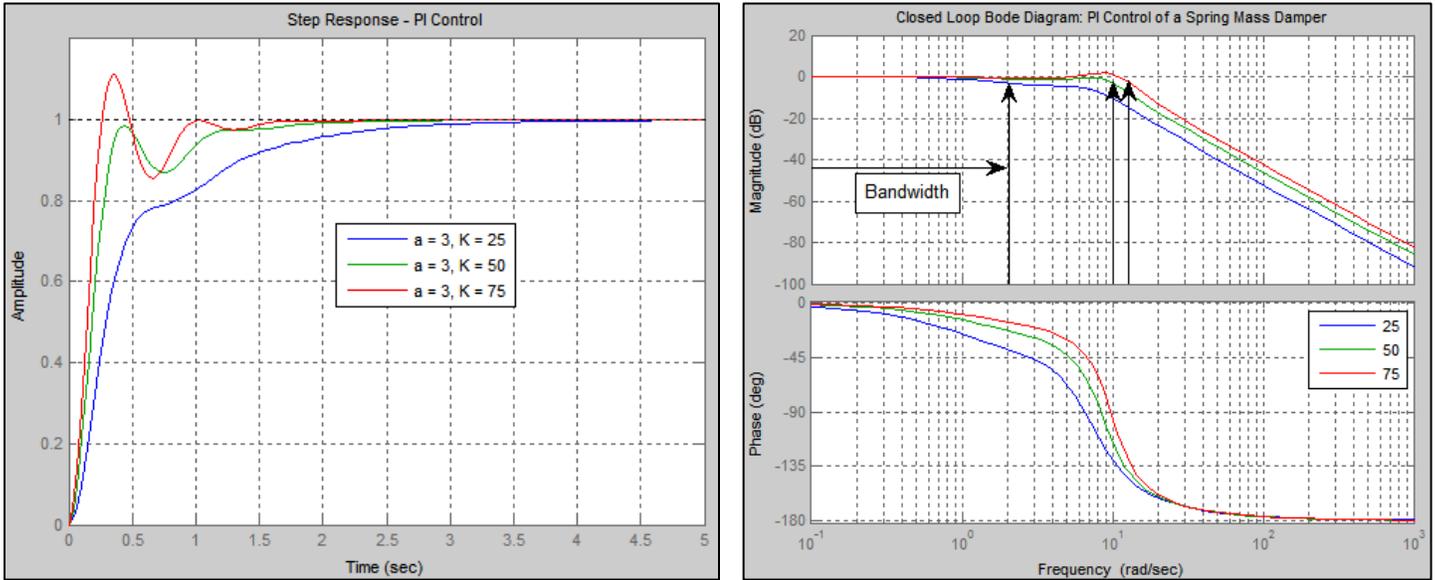
Fig. 5. Closed Loop Step Response and Bode Diagrams for PI Control $(a = 3)$

Proportional-Derivative (PD) Control

If **proportional-derivative (PD) control** is used, then

$$G_c(s) = K_p + K_D s = K_D(s + a) \qquad (6)$$

The parameters K_P and K_D represent the **proportional** and **derivative** gains, and the parameter $a = K_P/K_D$ is the **ratio** of the proportional and derivative gains. The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_D(s + a)}{s^2 + 8.8s + 40} \qquad \frac{X}{X_d}(s) = \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \qquad (7)$$

Without the integral control, this is again a **type-zero** system, and hence will have a **finite steady-state error** to a unit step input. Using the final-value theorem and the closed-loop transfer function, x_{ss} the final value of $x(t)$ to a unit step command is

$$x_{ss} = \lim_{s \to 0} \left(\cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \right) = \frac{K_D a}{40 + K_D a} = \frac{K_P}{40 + K_P} < 1 \qquad (8)$$

As with simple proportional control, the **larger the proportional gain**, the **smaller the steady-state error**.

The root locus diagram for the closed-loop system (with $a = 10$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in **Fig. 6**. The root locus diagram also shows the locations of the

closed-loop poles for a derivative gain $K_D \approx 25.6$. As the gain is **increased** the system's poles become **faster** and **more damped**. The Bode diagram indicates that the phase margin never drops below 90 degrees indicating a **very stable** closed-loop system for any gain.

Fig. 7 shows step responses and Bode diagrams of the closed loop system for $a = 10$ and derivative gains of $K_D = 10$, 27, 50, and 75. The PD controller has **decreased** the **system's settling time** considerably. However, to control the steady-state error, the derivative gain K_D must be high. This will **decrease** the **response times** and **increase** the **bandwidth** of the system and may make it **susceptible** to **noise**.
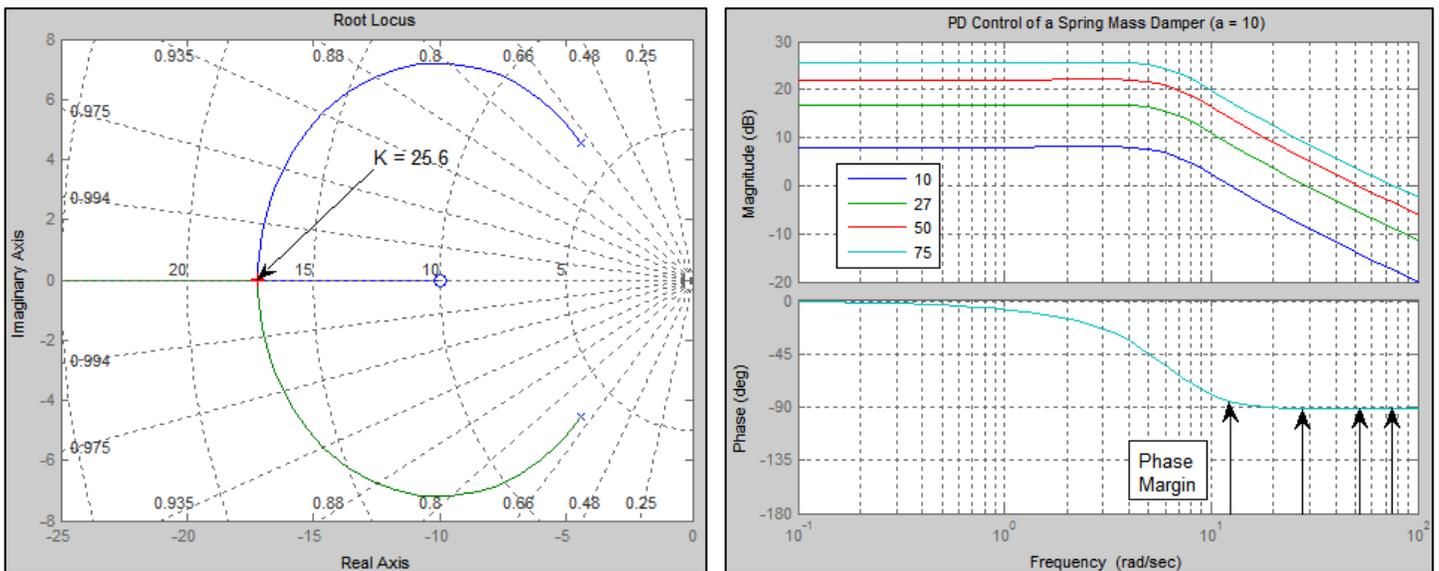

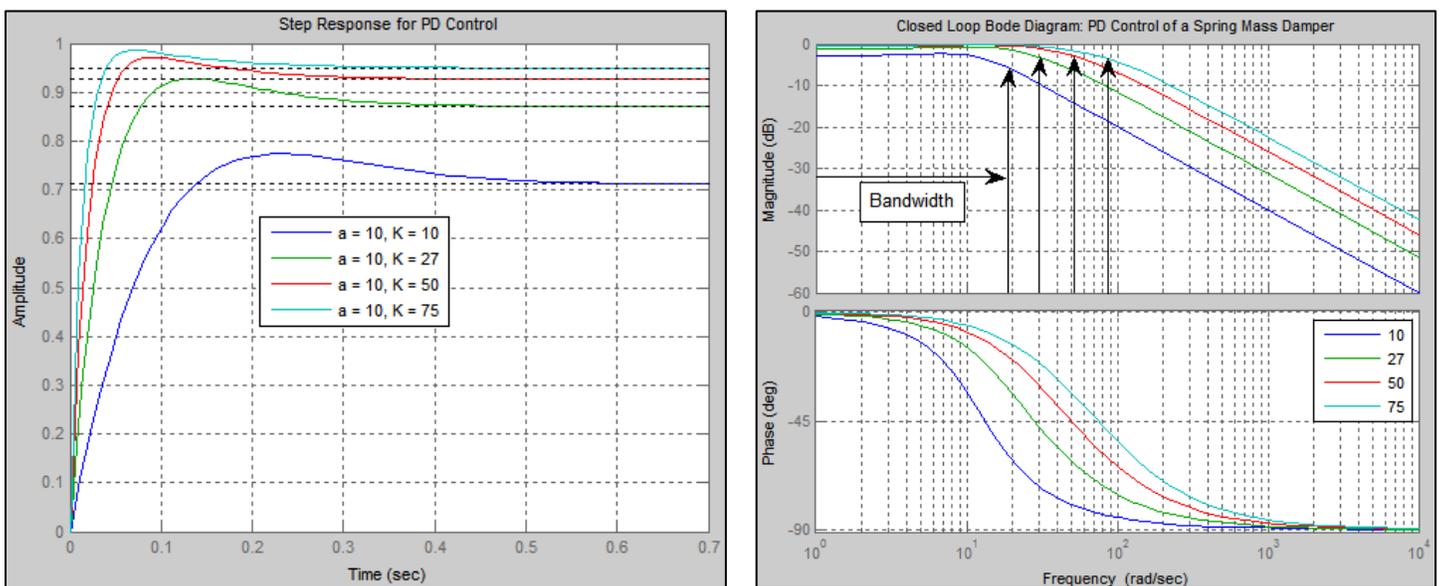Fig. 6. Root Locus Diagram and Bode Diagram for $(GH(s))$ for PD Control $(a = 10)$

Fig. 7. Closed Loop Step Response and Bode Diagrams for PD Control $(a = 10)$

Proportional-Integral-Derivative Control

If **proportional-integral-derivative (PID) control** is used, then

$$G_c(s) = K_p + \frac{K_I}{s} + K_D s = \frac{K_D(s^2 + as + b)}{s} \tag{9}$$

The parameters K_P, K_I, and K_D represent the **proportional**, **integral**, and **derivative** gains, the parameter $a = K_P/K_D$ is the **ratio** of the proportional and derivative gains, and the parameter $b = K_I/K_D$ is the **ratio** of the integral and derivative gains.

The loop and closed-loop transfer functions for this system are

$$GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)} \qquad \frac{X}{X_d}(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)} \tag{10}$$

Integral control makes the system **type-one**, so it has **zero** steady-state error for a step input.

The root locus diagram for the closed-loop system (with $a = 15$ and $b = 50$) for $K \geq 0$ and the Bode diagram for $GH(s)$ are shown in **Fig. 8**. The locations of the closed-loop poles for $K_D \approx 15.8$ are also shown. As the gain is **increased**, the system becomes **faster without significant losses** in the **phase margin**.
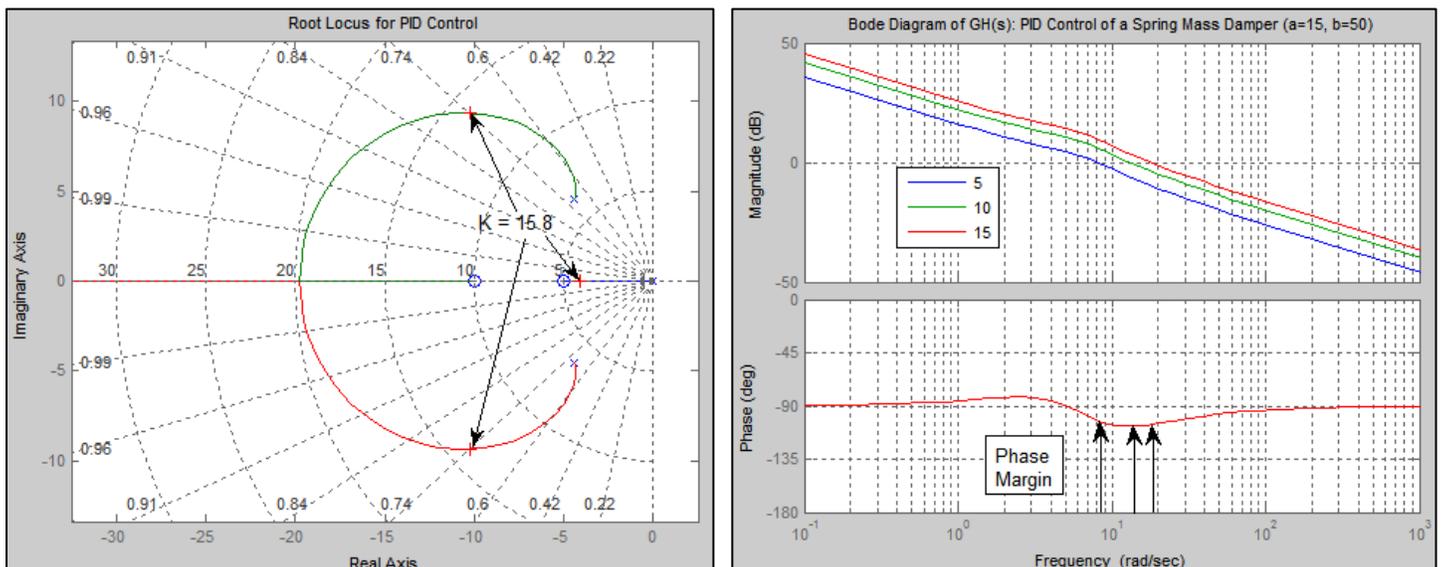

Fig. 8. Root Locus Diagram and Bode Diagram for ($GH(s)$) for PID Control ($a = 15, b = 50$)

Fig. 9 shows step responses and Bode diagrams of the closed-loop system for $a = 15$, $b = 50$, and derivative gains of $K_D = 5$, 10, and 15. Using both integral and derivative control has **removed**

steady-state error and ***decreased system settling times*** while maintaining a reasonable transient response.
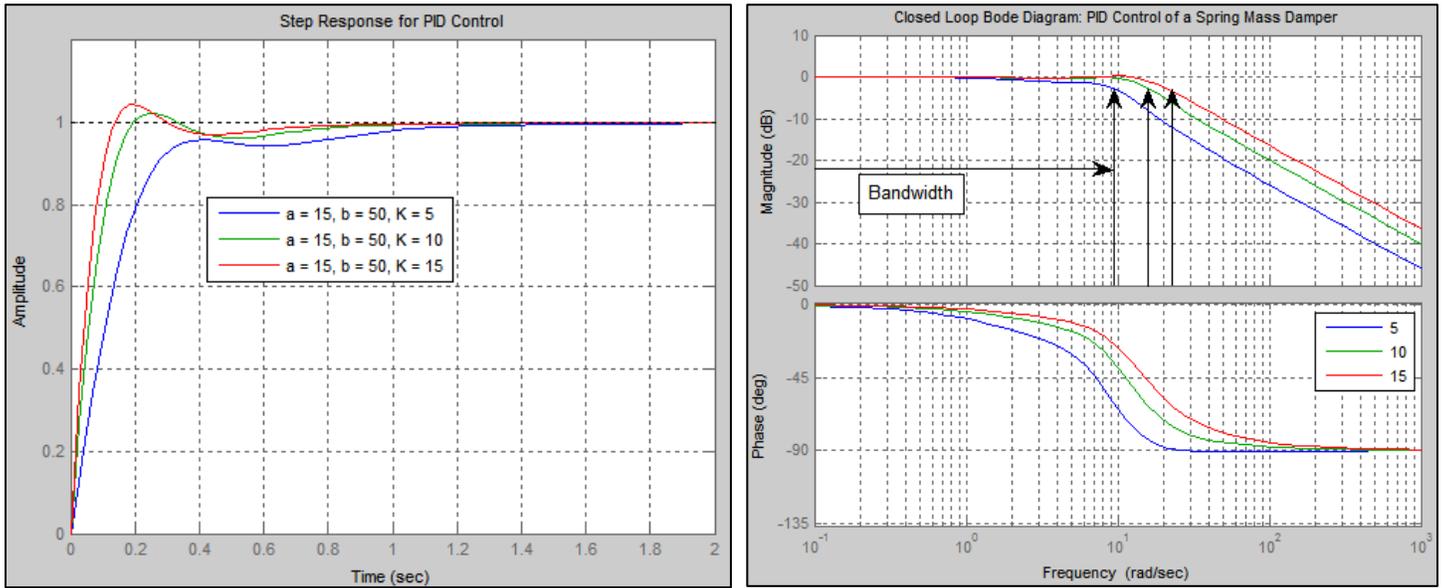

Fig. 9. Closed Loop Step Response and Bode Diagrams for PID Control $(a = 15, b = 50)$