

ME 4710 Motion and Control

Difference Equations and Approximate Solutions to Ordinary Differential Equations

Background

- Numerical methods can be used to *convert ordinary differential equations (ODE's)* into *difference equations*. These difference equations can be used to generate *approximate solutions* to the original ODE.
- *Many numerical methods exist* for this purpose, and generally they may be used to solve *linear* or *non-linear* ODE's. The solutions are a series of (discrete) values that approximate the solution.
- The *accuracy* of the approximate solution depends on the *method* used and the *increment* between successive values.
- In the notes that follow, it is assumed that the *independent variable* is *time* t , and that the sequence of values that represent the approximate solution will be *equally spaced* in time. The time increment between successive values is represented by T .
- The sequence of $N + 1$ time values at which the solution is approximated can then be written as $t = (0, T, 2T, \dots, kT, \dots, NT)$.

Euler's Method: Single, First-Order ODE

- Consider a first order ODE given by the equation

$$\dot{x} = f(x, t) \quad (1)$$

- Here x is the unknown variable, and f is a function of the unknown variable and time. To find an approximate solution to the differential equation, we expand the unknown $x(t)$ in a *Taylor series* about an arbitrary time $t = kT$.
- In *Euler's method*, the terms of order T^2 and higher are assumed to be small and are ignored.

$$\begin{aligned} x((k+1)T) &= x(kT) + T \left. \frac{dx}{dt} \right|_{t=kT} + \frac{1}{2} T^2 \left. \frac{d^2x}{dt^2} \right|_{t=kT} + \frac{1}{3!} T^3 \left. \frac{d^3x}{dt^3} \right|_{t=kT} + \dots \\ &\approx x(kT) + T \left. \frac{dx}{dt} \right|_{t=kT} \end{aligned} \quad (2)$$

- Using the differential equation to calculate the derivative of x in the truncated Taylor series gives the *difference* equation

$$\boxed{x((k+1)T) = x(kT) + (T \cdot f(x(kT), kT))} \quad (3)$$

- Eq. (3) can now be used to produce a sequence of values that approximate the solution to Eq. (1). The solution is second order accurate over a single step; however, it is only **first-order accurate** over **many steps**. Consequently, very small time steps are often required to produce reasonably accurate results.
- It should be noted that Eq. (3) predicts the value of x at time $t = (k + 1)T$ based on the value of x and its derivative \dot{x} at time $t = kT$. This is based on the idea that \dot{x} at time $t = kT$ can be approximated using a **first-order, forward difference**. That is,

$$\left. \frac{dx}{dt} \right|_{t=kT} \approx \frac{x((k + 1)T) - x(kT)}{T} \quad (4)$$

- A **modified Euler method** has been developed that predicts the value of x at time $t = (k + 1)T$ based on the value of x at time $t = kT$ and the **average** of \dot{x} at times $t = kT$ and $t = (k + 1)T$. Since the value of \dot{x} at time $t = (k + 1)T$ is not known, it is also estimated as part of the method. The details are not included here.

Runge-Kutta Method: Single, First-Order ODE

- **One** of the **most popular methods** for generating an approximate solution to Eq (1) is the Runge-Kutta method. Runge-Kutta methods of various levels of accuracy have been developed.
- The **second-order** Runge-Kutta method uses the following **difference** equations

$$x((k + 1)T) = x(kT) + a\Delta_1 + b\Delta_2 \quad (5)$$

where

$$\begin{aligned} \Delta_1 &= T \cdot f(x(kT), kT) \\ \Delta_2 &= T \cdot f(x(kT) + \beta\Delta_1, kT + \alpha T) \end{aligned} \quad (6)$$

- a , b , α , and β are **all constants**. Equating this expansion to a Taylor series expansion through the T^2 terms gives the following three equations that must be satisfied by the four unknown constants.

$$a + b = 1 \quad \alpha b = \frac{1}{2} \quad \beta b = \frac{1}{2} \quad (7)$$

- Using the values $a = b = \frac{1}{2}$ and $\alpha = \beta = 1$ provides the **same equations used for the modified Euler method** mentioned above. This method is third-order accurate for a single step and **second-order accurate for many steps**.
- **One** of the **most popular methods** used is a **fourth-order, Runge-Kutta method**. A commonly used set of **difference** equations used for this method are

$$x((k+1)T) = x(kT) + \frac{1}{6}(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) \quad (8)$$

where

$$\begin{aligned} \Delta_1 &= T \cdot f(x(kT), kT) \\ \Delta_2 &= T \cdot f(x(kT) + \frac{1}{2}\Delta_1, kT + \frac{1}{2}T) \\ \Delta_3 &= T \cdot f(x(kT) + \frac{1}{2}\Delta_2, kT + \frac{1}{2}T) \\ \Delta_4 &= T \cdot f(x(kT) + \Delta_3, kT + T) \end{aligned} \quad (9)$$

This method is fifth-order accurate for one step but only **fourth-order accurate** for **many steps**.

Multiple, First-Order ODE's

- Consider a set of m first-order, ODE's given by the equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \quad (10)$$

- Here \underline{x} represents an $m \times 1$ vector of unknown variables and \underline{f} represents an $m \times 1$ vector of functions of the unknowns and time. An **approximate solution** for each of the unknowns may be found by expanding each of the m unknowns $x_i(t)$ in a Taylor series about an arbitrary time $t = kT$.
- Using the results presented in Eq. (3), the difference equations for a **first-order, Euler approximations** are

$$\boxed{x_i((k+1)T) = x_i(kT) + (T \cdot f_i(\underline{x}(kT), kT))} \quad (i=1, \dots, m) \quad (11)$$

- Using the results presented in Equations (8) and (9), the difference equations for **fourth-order, Runge-Kutta approximations** are

$$\boxed{x_i((k+1)T) = x_i(kT) + \frac{1}{6}(\Delta_{i1} + 2\Delta_{i2} + 2\Delta_{i3} + \Delta_{i4})} \quad (12)$$

where

$$\boxed{\begin{aligned} \Delta_{i1} &= T \cdot f_i(\underline{x}(kT), kT) \\ \Delta_{i2} &= T \cdot f_i(\underline{x}(kT) + \frac{1}{2}\Delta_{i1}, kT + \frac{1}{2}T) \\ \Delta_{i3} &= T \cdot f_i(\underline{x}(kT) + \frac{1}{2}\Delta_{i2}, kT + \frac{1}{2}T) \\ \Delta_{i4} &= T \cdot f_i(\underline{x}(kT) + \Delta_{i3}, kT + T) \end{aligned}} \quad (i=1, \dots, m) \quad (13)$$

Here, Δ_j ($j=1,2,3$) are $m \times 1$ vectors whose elements are Δ_{ij} ($i=1, \dots, m$).

Example 1: $\frac{dx}{dt} + ax = u(t)$ with $x(0) = 0$ and $a = \text{constant}$. (*1st order, linear ODE*)

Euler's Method

$$x((k+1)t) = x(kT) + T \cdot f(x(kT), kT)$$

or

$$x((k+1)T) = x(kT) + T[u(kT) - ax(kT)] = (1 - aT)x(kT) + Tu(kT) \tag{14}$$

Fourth-Order Runge-Kutta

$$x((k+1)T) = x(kT) + \frac{1}{6}(\Delta_1 + 2\Delta_2 + 2\Delta_3 + \Delta_4) \tag{15}$$

where

$$\begin{aligned} \Delta_1 &= T[u(kT) - ax(kT)] \\ \Delta_2 &= T[u(kT + \frac{1}{2}T) - a(x(kT) + \frac{1}{2}\Delta_1)] \\ \Delta_3 &= T[u(kT + \frac{1}{2}T) - a(x(kT) + \frac{1}{2}\Delta_2)] \\ \Delta_4 &= T[u((k+1)T) - a(x(kT) + \Delta_3)] \end{aligned} \tag{16}$$

Results

- Letting the parameter $a = 5$, the *input* $u(t) = u_s(t)$ (the *unit step* function), and the time step $T = 0.1$ (sec), the two methods provide the results shown in *Fig 1*.
- Note that even with a fairly large time step, the *Runge-Kutta* approximation *matches* the exact solution very closely.

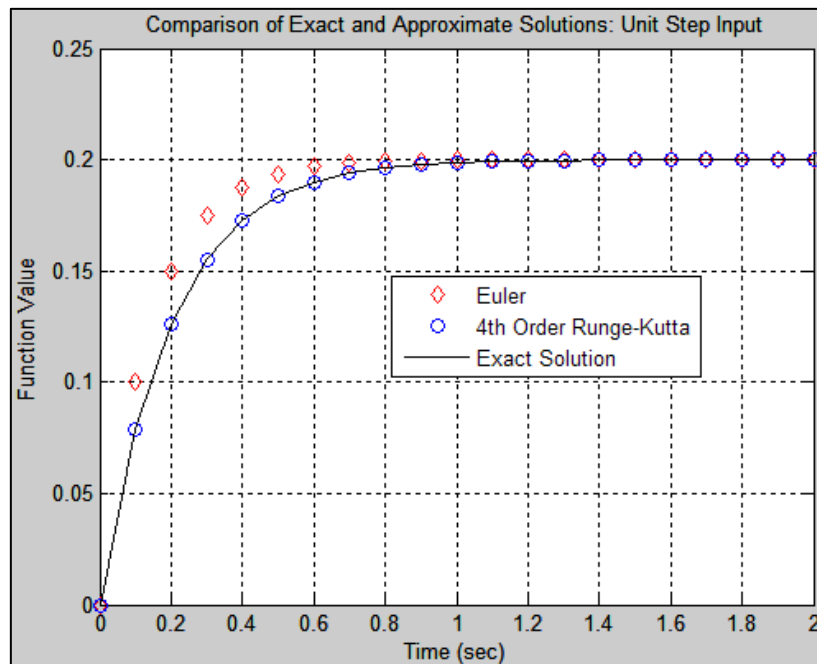


Fig 1. Comparison of Exact and Approximate Results – Unit Step Input

- Letting the parameter $a = 5$, the **input** $u(t) = 32e^{-3t} \cos(2t)$, and the time step $T = 0.02$ (sec), the two methods provide the results shown in **Fig 2**.
- With a more **active input**, a smaller time step is required to generate accurate results. Both methods provide reasonably accurate results; however, the Euler approximation over-predicts the peak response while the Runge-Kutta method under-predicts it.

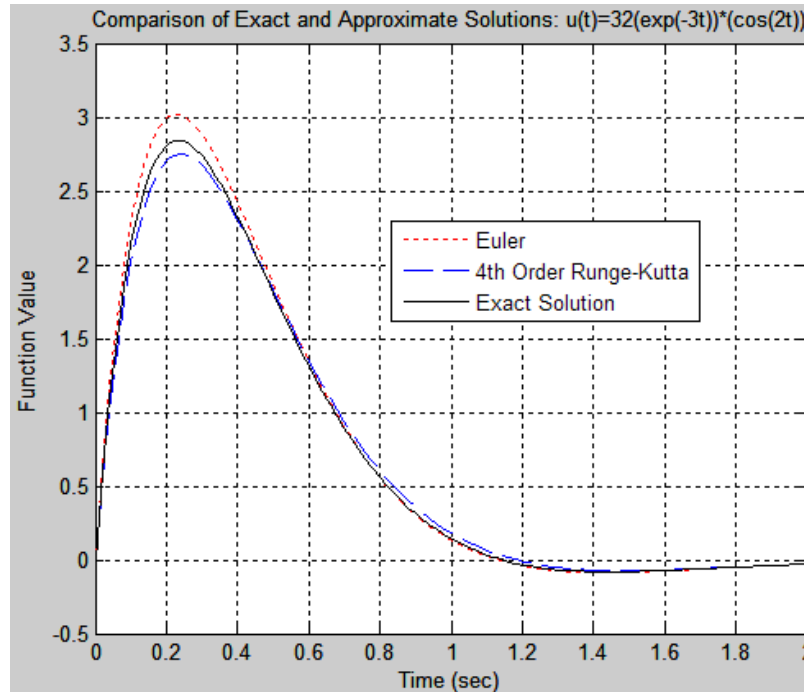


Fig 2. Comparison of Exact and Approximate Results: $u(t) = 32e^{-3t} \cos(2t)$

Example 2: $\frac{d^2\theta}{dt^2} + \left(\frac{g}{\ell}\right) \sin(\theta) = 0$ with $\theta(0) = \pi/4$ and $\dot{\theta}(0) = 0$. (**2nd order, non-linear**)

- To solve a second-order ODE, the equation must first be written as two first-order ODE's. To do this, let $x = \theta$ and $y = \dot{\theta} = \dot{x}$. Using the variables x and y , the original ODE can be written as

$$\begin{cases} \dot{x} = y = f_1(x, y, t) \\ \dot{y} = -(g/\ell) \sin(x) = f_2(x, y, t) \end{cases} \text{ with } x(0) = \pi/4 \text{ and } y(0) = 0 \quad (17)$$

Euler's Method

$$x((k+1)T) = x(kT) + T \cdot f_1(x(kT), y(kT), kT)$$

$$y((k+1)T) = y(kT) + T \cdot f_2(x(kT), y(kT), kT)$$

or

$$\begin{aligned} x((k+1)T) &= x(kT) + T y(kT) \\ y((k+1)T) &= y(kT) - T(g/\ell)\sin(x(kT)) \end{aligned} \quad (18)$$

Fourth-Order Runge-Kutta

$$\begin{aligned} x((k+1)T) &= x(kT) + \frac{1}{6}(\Delta_{11} + 2\Delta_{12} + 2\Delta_{13} + \Delta_{14}) \\ y((k+1)T) &= y(kT) + \frac{1}{6}(\Delta_{21} + 2\Delta_{22} + 2\Delta_{23} + \Delta_{24}) \end{aligned} \quad (19)$$

where

$$\begin{aligned} \Delta_{11} &= T \cdot f_1(\underline{x}(kT), kT) = T y(kT) \\ \Delta_{21} &= T \cdot f_2(\underline{x}(kT), kT) = -T(g/\ell)\sin(x(kT)) \\ \Delta_{12} &= T \cdot f_1(\underline{x}(kT) + \frac{1}{2}\underline{\Delta}_1, kT + \frac{1}{2}T) = T(y(kT) + \frac{1}{2}\Delta_{21}) \\ \Delta_{22} &= T \cdot f_2(\underline{x}(kT) + \frac{1}{2}\underline{\Delta}_1, kT + \frac{1}{2}T) = -T(g/\ell)\sin(x(kT) + \frac{1}{2}\Delta_{11}) \\ \Delta_{13} &= T \cdot f_1(\underline{x}(kT) + \frac{1}{2}\underline{\Delta}_2, kT + \frac{1}{2}T) = T(y(kT) + \frac{1}{2}\Delta_{22}) \\ \Delta_{23} &= T \cdot f_2(\underline{x}(kT) + \frac{1}{2}\underline{\Delta}_2, kT + \frac{1}{2}T) = -T(g/\ell)\sin(x(kT) + \frac{1}{2}\Delta_{12}) \\ \Delta_{14} &= T \cdot f_1(\underline{x}(kT) + \underline{\Delta}_3, kT + T) = T(y(kT) + \Delta_{23}) \\ \Delta_{24} &= T \cdot f_2(\underline{x}(kT) + \underline{\Delta}_3, kT + T) = -T(g/\ell)\sin(x(kT) + \Delta_{13}) \end{aligned} \quad (20)$$

Results

- Letting $g = 9.81 \text{ (m/s}^2\text{)}$, the length $\ell = 0.5 \text{ (m)}$, and the time step $T = 0.01 \text{ (sec)}$, the two methods produce the results shown in **Fig 3** for the angle of the pendulum. Since no damping was included, the *pendulum* should *continue to oscillate* between $\pm \frac{\pi}{4} = \pm 0.7854 \text{ (rad)}$.
- Clearly, the ***Euler method is unstable*** at this time step. Conversely, the ***Runge-Kutta method*** appears to be ***tracking*** the solution ***very well***.
- **Fig 4** shows results for the two methods for a ***smaller time step*** $T = 0.001 \text{ (sec)}$. The ***Euler method*** is producing much better results at this time step. However, after four periods, it is evident that the oscillations are beginning to increase in amplitude, indicating that the method ***may*** still be ***unstable***.

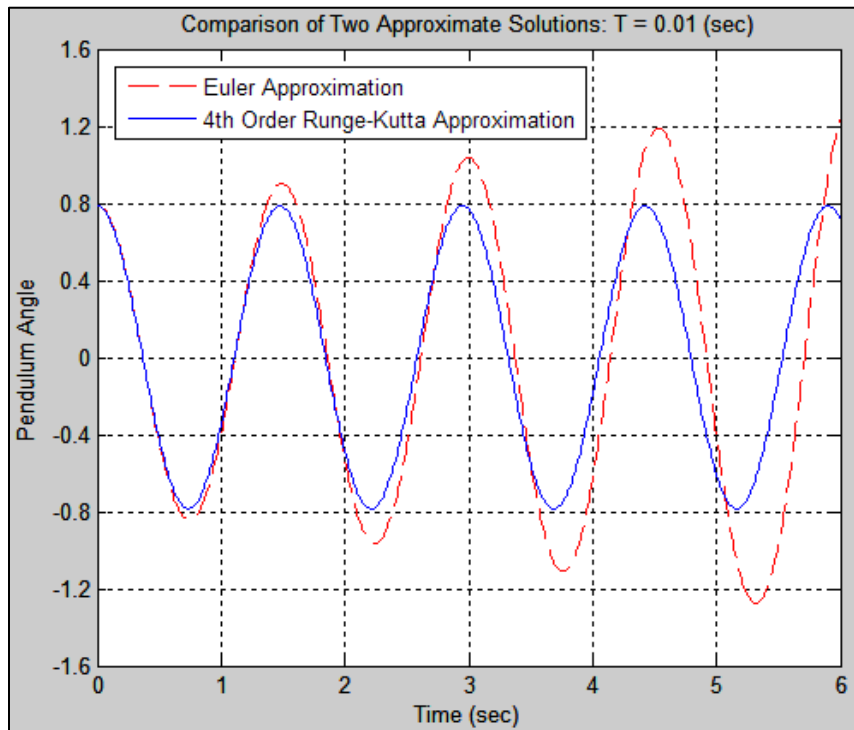


Figure 3.
Euler & Runge-Kutta
Approximations to
Pendulum Equation
($T=0.01$ (sec))

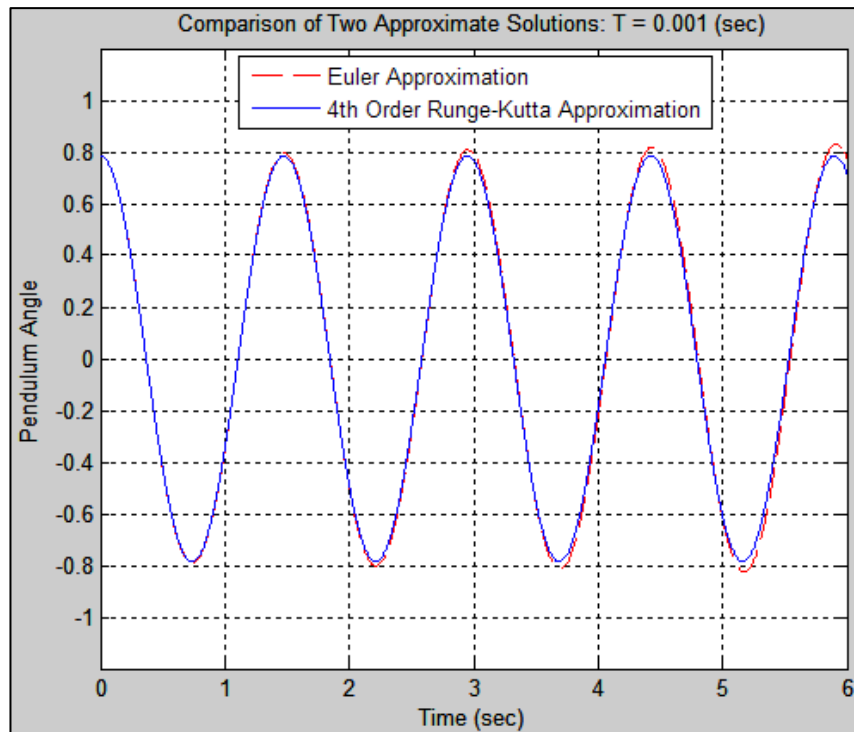


Figure 4.
Euler & Runge-Kutta
Approximations to
Pendulum Equation
($T=0.001$ (sec))