

**ME 4710 Motion and Control**  
**Z Transforms and the Zero-order-hold (ZOH)**

The *Laplace transform* is used to find *transfer functions* for *continuous systems*. The *Z-transform* is used to find *discrete transfer functions* for systems. The systems themselves may be *continuous* or *discrete*.

Consider a discretized function based on sampled data

$$y_d(t) = \sum_{k=0}^{\infty} y(kT)\delta(t - kT) \quad \text{where} \quad \delta(t - kT) = \begin{cases} 0 & \text{for } t \neq kT \\ 1 & \text{for } t = kT \end{cases}$$

Taking Laplace transforms of the above equation gives

$$\mathcal{L}(y_d(t)) = \sum_{k=0}^{\infty} y(kT)\mathcal{L}(\delta(t - kT)) = \sum_{k=0}^{\infty} y(kT)e^{-skT}$$

where  $e^{-skT}$  represents a *time delay* of  $kT$  seconds. Defining the variable  $z = e^{sT}$ , the Z-transform of a sequence of numbers (e.g. sampled data)  $y(kT)$  is defined as

$$Z(y(kT)) = Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

This is an *infinite series* in the variable  $z$ . The term  $y(kT)z^{-k}$  indicates the value  $y(kT)$  occurs at time  $t = kT$ , a *delay* of  $kT$  seconds from the *start* of the sequence.

Like the *Laplace transform*, the *Z transform* is a *linear* transformation. *One* important *property* of the Z transform provides the means to find *difference equations* associated with a discrete transfer function. That is,

$$Z(y(k - 1)) = z^{-1}Z(y(k)) = z^{-1}Y(z) \quad \Rightarrow \quad Z^{-1}(z^{-1}Y(z)) = y(k - 1)$$

Here, the condensed notation  $y(k) \triangleq y(kT)$  is used for simplicity.

A *second* useful *property*, the *discrete final value theorem* states

$$\lim_{k \rightarrow \infty} (y(k)) = \lim_{z \rightarrow 1} \left( (1 - z^{-1})Y(z) \right)$$

The continuous and discrete final value theorems can be used to *verify* that the *final value* produced by a continuous transfer function and an equivalent discrete transfer function are the *same*.

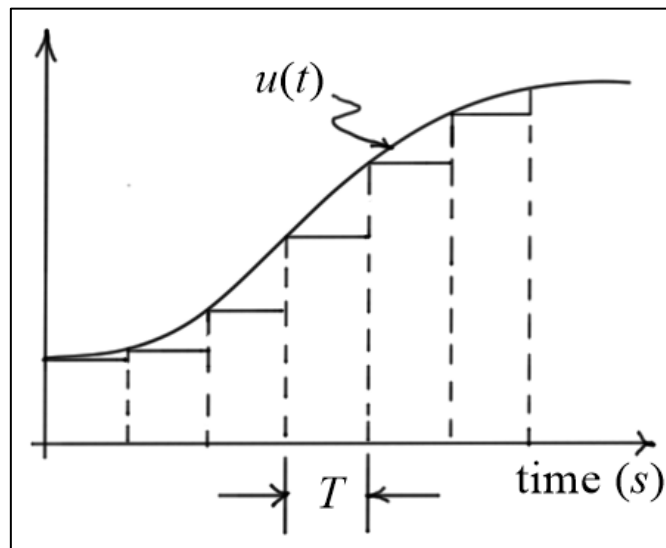
## Digital-to-Analog Conversion and Zero-Order Hold (ZOH)

At each sampling time, the digital-to-analog converter (DAC) *changes* a digital value into an analog signal by *sampling* the data and *holding* the signal at that level until the next sample time occurs. This is referred to as a *zero-order hold* and is represented by the transfer function

$$G_{ZOH}(s) = \frac{1}{s} - \frac{1}{s} e^{-sT} = \frac{1 - e^{-sT}}{s}$$

This transfer function represents the *difference* of *two step functions*. The second step is delayed by  $T$  seconds (multiplication by  $e^{-sT}$ ), so the difference is a *square pulse* of *unit amplitude* lasting for  $T$  seconds.

When applied to the sequence  $u(kT)$  the output is a *series of square pulses* that represent  $u(t)$ . The *smaller* the sample period  $T$ , the *better* the characterization of  $u(t)$ .



Signal Representation Using a *Zero-Order Hold*