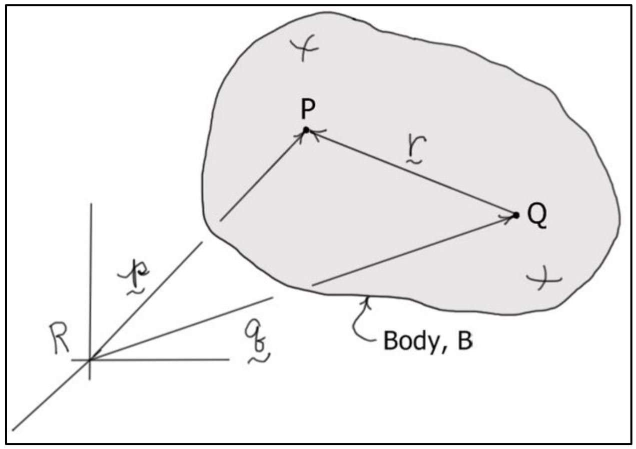
ME 5550 Intermediate Dynamics
Derivatives of a Vector in Two Different Reference Frames
("The Derivative Rule")

Motivation

It is often convenient to express vectors in terms of local (or rotating) unit vector sets (reference frames). For example, consider the position vector $\underset{\sim}{r}_{P/Q}$ the position vector of P relative to Q shown in the diagram. This vector describes the relative position of P and Q, two points ***fixed*** in the rigid body B. As such, it is most easily described in terms of the unit vector set $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ that is fixed in (and rotates with) *B*.



To describe the ***relative motion*** of P and Q, the position vector $\underset{\sim}{r}_{P/Q}$ must be differentiated in the fixed reference frame (unit vector set $R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$). This can be done in one of two ways:

1) express $\underset{\sim}{r}_{P/Q}$ in terms of $R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$, and then ***differentiate***, or

2) express $\underset{\sim}{r}_{P/Q}$ in terms of $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$, and then ***use*** the ***derivative rule***.

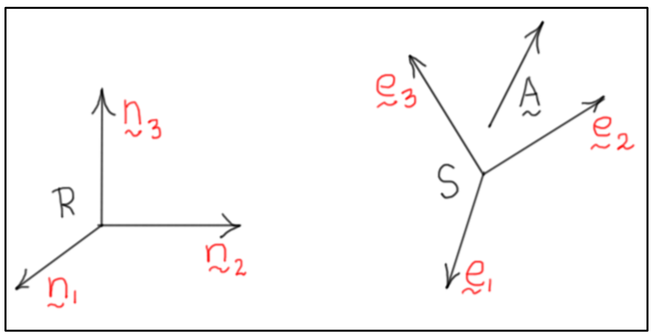
The Derivative Rule

Given the two reference frames

$R:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ (rotating frame)

$S:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ (rotating frame),

the derivatives of ***any*** vector $\underset{\sim}{A}$ in the two reference frames are related by the following rule



$$\frac{{}^R d\underset{\sim}{A}}{dt} = \frac{{}^S d\underset{\sim}{A}}{dt} + ({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{A})$$

Here, ${}^R\underset{\sim}{\omega}_S$ is the angular velocity of frame *S* relative to the frame *R*.

Derivation

Consider the vector $\underset{\sim}{A}$ and the two reference frames R and S as shown. Suppose, for convenience, $\underset{\sim}{A}$ is expressed in terms of the unit vectors of frame S. That is,

$$\boxed{\underset{\sim}{A} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3}$$

Then, the derivative of $\underset{\sim}{A}$ in the reference frame R can be computed as follows

$$\begin{aligned}
\frac{{}^R d\underset{\sim}{A}}{dt} &= \underbrace{(\dot{a}_1 \underset{\sim}{e}_1 + \dot{a}_2 \underset{\sim}{e}_2 + \dot{a}_3 \underset{\sim}{e}_3)} + a_1 \frac{{}^R d\underset{\sim}{e}_1}{dt} + a_2 \frac{{}^R d\underset{\sim}{e}_2}{dt} + a_3 \frac{{}^R d\underset{\sim}{e}_3}{dt} \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + a_1({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_1) + a_2({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_2) + a_3({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_3) \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + {}^R\underset{\sim}{\omega}_S \times (a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3) \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + ({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{A})
\end{aligned}$$

Here ${}^R\underset{\sim}{\omega}_S$ is the angular velocity of the reference frame $S:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ relative to the reference frame $R:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$.