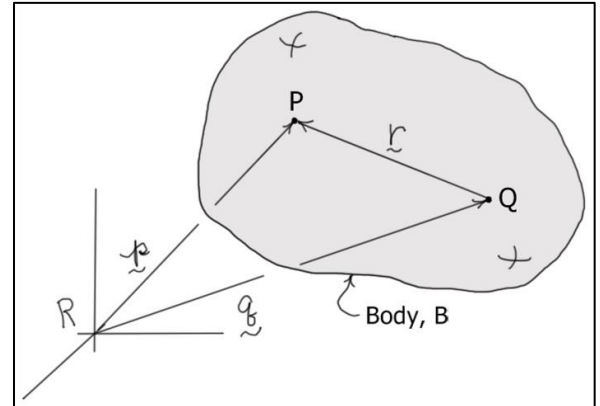


ME 5550 Intermediate Dynamics

Relative Kinematics of Two Points Fixed on a Rigid Body

General Concept

Consider the three-dimensional motion of a rigid body B as shown in the diagram. The points P and Q represent two points that are fixed in the body. The velocities and accelerations of P and Q in the reference frame R are related as follows



$$\underline{v}_P = \underline{v}_Q + \underline{v}_{P/Q} = \underline{v}_Q + (\underline{\omega}_B \times \underline{r})$$

and

$$\underline{a}_P = \underline{a}_Q + \underline{a}_{P/Q} = \underline{a}_Q + (\underline{\alpha}_B \times \underline{r}) + \underline{\omega}_B \times (\underline{\omega}_B \times \underline{r})$$

These equations are easily verified using “the derivative rule” discussed in previous notes.

Derivation

1. The position vector of point P relative to the reference frame R can be written as $\underline{p} = \underline{q} + \underline{r}$.

Differentiating this equation and using “the derivative rule” gives

$$\begin{aligned} \underline{v}_P &= \frac{d\underline{p}}{dt} = \frac{d\underline{q}}{dt} + \frac{d\underline{r}}{dt} \\ &= \underline{v}_Q + \frac{d\underline{r}}{dt} + \underline{\omega}_B \times \underline{r} \quad \Rightarrow \quad \underline{v}_P = \underline{v}_Q + \underline{v}_{P/Q} = \underline{v}_Q + (\underline{\omega}_B \times \underline{r}_{P/Q}) \\ &= \underline{v}_Q + \underline{\omega}_B \times \underline{r} \end{aligned}$$

Here,

$$\frac{d\underline{r}}{dt} = \underline{v}_{P/Q} = \underline{\omega}_B \times \underline{r}_{P/Q} \quad (\text{velocity of } P \text{ relative to } Q \text{ in } R, \underline{v}_{P/Q})$$

2. Differentiating the velocity equation and again using “the derivative rule” gives

$$\begin{aligned} \underline{a}_P &= \frac{d}{dt}(\underline{v}_P) = \frac{d}{dt}(\underline{v}_Q) + \frac{d}{dt}(\underline{\omega}_B \times \underline{r}) \\ &= \underline{a}_Q + \left(\frac{d}{dt}(\underline{\omega}_B) \times \underline{r} \right) + \left(\underline{\omega}_B \times \frac{d\underline{r}}{dt} \right) \\ &= \underline{a}_Q + (\underline{\alpha}_B \times \underline{r}) + \underline{\omega}_B \times (\underline{\omega}_B \times \underline{r}) \end{aligned}$$

or

$$\boxed{{}^R \underline{a}_P = {}^R \underline{a}_Q + {}^R \underline{a}_{P/Q} = {}^R \underline{a}_Q + ({}^R \underline{\alpha}_B \times \underline{r}) + {}^R \underline{\omega}_B \times ({}^R \underline{\omega}_B \times \underline{r})}$$

Here ${}^R \underline{a}_{P/Q}$ is the acceleration of P with respect to Q in R, and by inspection of the above equation, it is defined to be

$$\boxed{{}^R \underline{a}_{P/Q} = \frac{{}^R d}{{}^R dt} ({}^R \underline{v}_{P/Q}) = ({}^R \underline{\alpha}_B \times \underline{r}) + {}^R \underline{\omega}_B \times ({}^R \underline{\omega}_B \times \underline{r})}$$

Notes

1. The above formulae may be applied *recursively* to calculate motions of *remote points* within a mechanical system.
2. Calculation of velocities and accelerations *does not require differentiation*, only multiplication and addition.