

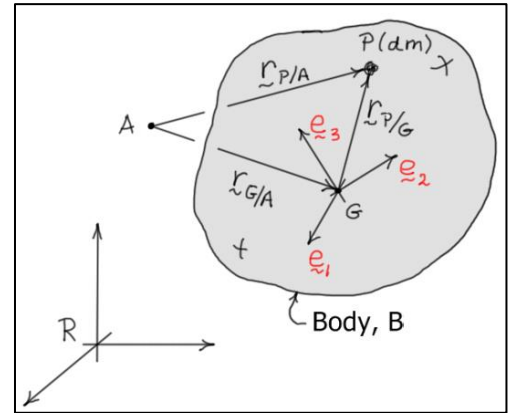
ME 5550 Intermediate Dynamics

Angular Momentum of a Rigid Body about an Arbitrary Point

The *angular momentum* of a rigid body B about its mass center G is defined as

$$\boxed{\underline{H}_G = \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_P) dm} \quad (1)$$

In previous notes, it was shown that \underline{H}_G can be written in terms of the body's moments and products of inertia and the body-fixed angular velocity components as follows.



$$\boxed{\underline{H}_G = (I_{xx}^G \omega_1 - I_{xy}^G \omega_2 - I_{xz}^G \omega_3) \underline{n}_1 + (-I_{xy}^G \omega_1 + I_{yy}^G \omega_2 - I_{yz}^G \omega_3) \underline{n}_2 + (-I_{xz}^G \omega_1 - I_{yz}^G \omega_2 + I_{zz}^G \omega_3) \underline{n}_3}$$

or

$$\boxed{\underline{H}_G = \underline{I}_G \cdot {}^R \underline{\omega}_B} \quad (2)$$

Here, \underline{I}_G represents the *inertia dyadic* (or matrix) of B about its mass-center G , and ${}^R \underline{\omega}_B$ represents the *angular velocity* of the body.

The angular momentum of a rigid body about an *arbitrary point* A is similarly defined as

$$\boxed{\underline{H}_A = \int_B (\underline{r}_{P/A} \times {}^R \underline{v}_P) dm} \quad (3)$$

Substituting $\underline{r}_{P/A} = \underline{r}_{G/A} + \underline{r}_{P/G}$ into the integrand, distributing the cross product over the sum, and expanding gives

$$\begin{aligned} \underline{H}_A &= \int_B ((\underline{r}_{G/A} + \underline{r}_{P/G}) \times {}^R \underline{v}_P) dm = \int_B (\underline{r}_{G/A} \times {}^R \underline{v}_P) dm + \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_P) dm \\ &= \underline{r}_{G/A} \times \left(\int_B {}^R \underline{v}_P dm \right) + \int_B (\underline{r}_{P/G} \times {}^R \underline{v}_P) dm \\ &= \underline{r}_{G/A} \times (m {}^R \underline{v}_G) + \underline{H}_G \end{aligned}$$

Or,

$$\boxed{\underline{H}_A = \underline{H}_G + \underline{r}_{G/A} \times m {}^R \underline{v}_G} \quad (4)$$

So, the angular momentum of the body about an arbitrary point A is equal to the angular momentum of the body about its mass-center G plus the moment of the body's linear momentum about A .

Special Case: Motion about a Fixed-Point on the Body

If some point O of the rigid body is *fixed* so that the body *pivots* about this point, the velocity of the mass center may be written as ${}^R\mathcal{V}_G = {}^R\mathcal{V}_O + {}^R\omega_B \times \mathcal{r}_{G/O} = {}^R\omega_B \times \mathcal{r}_{G/O}$. Substituting this result into Eq. (4) above and combining terms, it can be shown that the angular momentum of the body about point O can be written as

$$\boxed{\mathcal{H}_O = \mathcal{I}_O \cdot {}^R\omega_B} \tag{5}$$

Here, \mathcal{I}_O represents the inertia dyadic (or matrix) about the fixed-point O .