ME 5550 Intermediate Dynamics
Angular Velocity and Orientation Angles

When using ***sequences*** of ***angles*** to describe the ***orientation*** of a rigid body, the ***summation rule*** for ***angular velocities*** is used to find the angular velocity of the body. Consider the case where a 1-2-3 rotation sequence is used to describe the orientation of a body. In that case, the angular velocity of the body can be written as

$$\boxed{{}^{R}\underset{\sim}{\omega}_{B} = {}^{R}\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_{B} = \dot{\theta}_1 \underset{\sim}{N}_1 + \dot{\theta}_2 \underset{\sim}{N}'_2 + \dot{\theta}_3 \underset{\sim}{N}''_3 = \dot{\theta}_1 \underset{\sim}{N}'_1 + \dot{\theta}_2 \underset{\sim}{N}''_2 + \dot{\theta}_3 \underset{\sim}{n}_3}$$

Recall that the "***primed***" axes are ***intermediate*** axes that are introduced so the angular velocity components are all "***simple***". To make the form of ${}^{R}\underset{\sim}{\omega}_{B}$ most useful, it must be expressed in either the base reference frame $\left(R: \underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ or in the body reference frame $\left(B: \underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$. For example, it is easy to show that the ***body-fixed components*** are

$$\boxed{\begin{aligned} &\omega_1 = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \\ &\omega_2 = -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \\ &\omega_3 = \dot{\theta}_1 S_2 + \dot{\theta}_3 \end{aligned}} \qquad \left({}^{R}\underset{\sim}{\omega}_{B} = \omega_1 \underset{\sim}{n}_1 + \omega_2 \underset{\sim}{n}_2 + \omega_3 \underset{\sim}{n}_3\right)$$

These equations may be ***inverted*** to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of ω_1, ω_2, and ω_3 to give

$$\boxed{\begin{aligned} &\dot{\theta}_1 = \left(\omega_1 C_3 - \omega_2 S_3\right) \big/ C_2 \\ &\dot{\theta}_2 = \omega_1 S_3 + \omega_2 C_3 \\ &\dot{\theta}_3 = \omega_3 - S_2\left(\omega_1 C_3 - \omega_2 S_3\right) \big/ C_2 \end{aligned}}$$

<u>Singularity Positions</u>

The angular velocity equations shown above for the 1-2-3 orientation angle sequence are ***singular*** when $C_2 \triangleq \cos(\theta_2)$ is ***zero***, that is, when the ***second orientation angle*** is 90 degrees. ***All orientation angle sequences display such a singularity*** at some position. This can cause problems for computer programs that use angle sequences to describe the orientation of rigid bodies.

Linearization of the 1-2-3 Orientation Angle Sequence about (0,0,0)

The above equations that relate the ***derivatives*** of the ***orientation angles*** to the ***angular velocity components*** were derived assuming the body is in an arbitrary position. Now, it will be assumed that the body is exhibiting ***small angular motions*** about the orientation where ***all angles*** are ***zero***, that is, around the equilibrium position $\theta_1 = \theta_2 = \theta_3 = \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$. In this case, the angular velocity of the body can be described using equations ***linearized*** about the ***equilibrium position***. The linearizations of the angular velocity expressions about the equilibrium position follow.

1. $f(\theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2) = \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \cong m_1\theta_2 + m_2\theta_3 + m_3\dot{\theta}_1 + m_4\dot{\theta}_2$

$$m_1 = \left.\frac{\partial f}{\partial \theta_2}\right|_{eq} = \left(-\dot{\theta}_1 S_2 C_3\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \theta_3}\right|_{eq} = (-\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(C_2 C_3\right)_{eq} = 1 \qquad m_4 = \left.\frac{\partial f}{\partial \dot{\theta}_2}\right|_{eq} = \left(S_3\right)_{eq} = 0$$

2. $g(\theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2) = -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \cong m_1\theta_2 + m_2\theta_3 + m_3\dot{\theta}_1 + m_4\dot{\theta}_2$

$$m_1 = \left.\frac{\partial g}{\partial \theta_2}\right|_{eq} = \left(\dot{\theta}_1 S_2 S_3\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \theta_3}\right|_{eq} = \left(-\dot{\theta}_1 C_2 C_3 - \dot{\theta}_2 S_3\right)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(-C_2 S_3\right)_{eq} = 0 \qquad m_4 = \left.\frac{\partial f}{\partial \dot{\theta}_2}\right|_{eq} = \left(C_3\right)_{eq} = 1$$

2. $h(\theta_2, \dot{\theta}_1, \dot{\theta}_3) = \dot{\theta}_1 S_2 + \dot{\theta}_3 \cong m_1\theta_2 + m_2\dot{\theta}_1 + m_3\dot{\theta}_3$

$$m_1 = \left.\frac{\partial g}{\partial \theta_2}\right|_{eq} = \left(\dot{\theta}_1 C_2\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(S_2\right)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_3}\right|_{eq} = 1$$

So, the ***final result*** of the ***linearization*** is that $\omega_1 \cong \dot{\theta}_1$, $\omega_2 \cong \dot{\theta}_2$, and $\omega_3 \cong \dot{\theta}_3$.