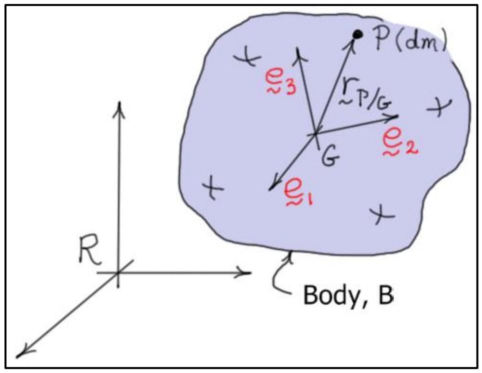
ME 5550 Intermediate Dynamics
Moments and Products of Inertia and the Inertia Matrix

Moments of Inertia

A rigid body B is shown in the diagram below. The unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are fixed in the body and are directed along a ***convenient*** set of axes (x, y, z) that pass through the mass center G. The ***moments of inertia*** of the body about these axes are defined as follows



$$I_{xx}^G = \int_B (y^2 + z^2)\, dm \qquad I_{yy}^G = \int_B (x^2 + z^2)\, dm \qquad I_{zz}^G = \int_B (x^2 + y^2)\, dm$$

Here, x, y, and z are defined as the $\underset{\sim}{e}_i \; (i = 1, 2, 3)$ components of $\underset{\sim}{r}_{P/G}$ the position vector of P with respect to G, that is, $\underset{\sim}{r}_{P/G} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3$.

Moments of inertia of a body about an axis ***measure the distribution*** of the ***body's mass about that axis***. The smaller the inertia the more the mass is concentrated about the axis. Inertia values can be found either by ***measurement*** or by ***calculation***. Calculations are based on ***direct integration*** and/or on the "***body build-up***" technique. In the body build-up technique, ***inertias of simple shapes are added*** to estimate the inertia of a composite shape. The inertias of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are transferred to axes through the composite mass center using the ***Parallel Axes Theorem for Moments of Inertia***.

Parallel Axes Theorem for Moments of Inertia

The inertia (I_i^A) of a body about an axis (i) through any point (A) is equal to the inertia (I_i^G) of the body about a parallel axis through the mass center G plus the mass (m) times the distance (d_i) between the two parallel axes squared.

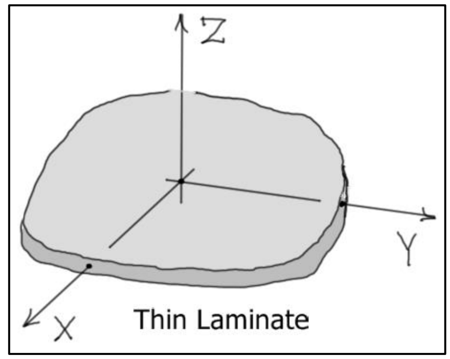
$$I_{ii}^A = I_{ii}^G + m\, d_i^2 \quad (i = x, y, \text{ or } z)$$

Note that moments of inertia are ***always positive***. From the parallel axes theorem, it is obvious that the ***minimum moments of inertia*** of a body occur about axes that pass through its ***mass center***.

Products of Inertia

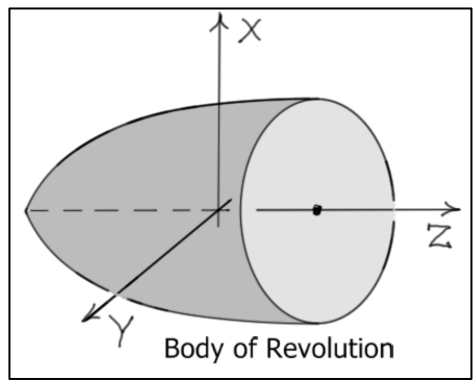
The ***products of inertia*** of the rigid body are defined as

$$I_{xy}^G = \int_B (xy)\, dm \qquad I_{xz}^G = \int_B (xz)\, dm \qquad I_{yz}^G = \int_B (yz)\, dm$$



The products of inertia of a body are measures of ***symmetry***. ***If a plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero***. For example, consider the ***thin laminate*** shown. The middle plane of the laminate lies in the XY-plane so that half its thickness is above the plane and half is below. Hence, the ***XY-plane*** is a ***plane of symmetry*** and

$$I_{xz} = I_{yz} = 0$$



Bodies of revolution have ***two planes*** of ***symmetry***. For the configuration shown, the XZ and YZ planes are planes of symmetry. Hence, ***all products of inertia are zero*** about the X, Y, and Z axes.

Products of inertia are found either by ***measurement*** or by ***calculation***. Calculations are based on ***direct integration*** or on the "***body build-up***" technique. In the body build-up technique, ***products of inertia of simple shapes are added*** to estimate the products of inertia of a composite shape. The products of inertia of simple shapes (about their individual mass centers) are found in ***standard inertia tables***. These values are transferred to axes through the composite mass center using the ***Parallel Axes Theorem for Products of Inertia***.

Parallel Axes Theorem for Products of Inertia

The product of inertia (I_{ij}^A) of a body about a pair of axes (i, j) passing through any point (A) is equal to the product of inertia (I_{ij}^G) of the body about a set of parallel axes through the mass center G plus the mass (m) times the product of the coordinates $(c_i c_j)$ of G relative to A (or A relative to G) measured along those axes.

$$I_{ij}^{A} = I_{ij}^{G} + m\, c_i\, c_j \qquad (i = x, y, \text{ or } z \text{ and } j = x, y, \text{ or } z)$$

Products of inertia may be ***positive***, ***negative***, or ***zero***.

The Inertia Matrix

The inertias of a body about a set of axes (passing through some point) are often collected into a single ***inertia matrix***. For example, the inertia matrix of a body about a set of axes through its mass center G is defined as

$$\left[I_G\right] = \begin{bmatrix} I_{11}^G & I_{12}^G & I_{13}^G \\ I_{21}^G & I_{22}^G & I_{23}^G \\ I_{31}^G & I_{32}^G & I_{33}^G \end{bmatrix} = \begin{bmatrix} I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\ -I_{xy}^G & I_{yy}^G & -I_{yz}^G \\ -I_{xz}^G & -I_{yz}^G & I_{zz}^G \end{bmatrix}$$

There is a ***different inertia matrix for each set of axes passing through a given point***. There is ***one set of directions*** for each point that ***renders the inertia matrix diagonal***. These directions are called ***principal directions*** (or ***principal axes***) of the body for that point. In general, the ***principal axes*** are ***different for each point*** in a body. Finally, note that ***all inertia matrices are symmetric***.

The Inertia Dyadic

The inertias of a body about a set of axes (passing through some point) may also be collected into a single ***inertia dyadic***. For example, the inertia dyadic of a body about a set of axes through its mass center G is defined as

$$\underset{\approx}{I}_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}^{G}\, \underset{\sim}{e}_i\, \underset{\sim}{e}_j$$

where I_{ij}^{G} $(i, j = 1,2,3)$ are the ***elements*** of the ***inertia matrix***, and the ***vector product*** $\underset{\sim}{e}_i \underset{\sim}{e}_j$ is called a ***dyad***.

The ***dot product*** of a dyad with a vector results in another vector. A dyad can be pre-dotted or post-dotted with a vector, and the results of the two operations are generally different. There are many ***properties*** that dyads satisfy. Three ***properties*** that are useful when using inertia dyadics are listed below.

1. $\underset{\sim}{c}\cdot\left(\underset{\sim}{a}\underset{\sim}{b}\right)=\left(\underset{\sim}{c}\cdot\underset{\sim}{a}\right)\underset{\sim}{b}$ and $\left(\underset{\sim}{a}\underset{\sim}{b}\right)\cdot\underset{\sim}{c}=\underset{\sim}{a}\left(\underset{\sim}{b}\cdot\underset{\sim}{c}\right)=\left(\underset{\sim}{b}\cdot\underset{\sim}{c}\right)\underset{\sim}{a}$ $\Rightarrow \boxed{\underset{\sim}{c}\cdot\left(\underset{\sim}{a}\underset{\sim}{b}\right)\neq\left(\underset{\sim}{a}\underset{\sim}{b}\right)\cdot\underset{\sim}{c}}$
2. $\left(\underset{\sim}{a}\underset{\sim}{b}+\underset{\sim}{c}\underset{\sim}{d}\right)\cdot\underset{\sim}{e}=\left(\underset{\sim}{b}\cdot\underset{\sim}{e}\right)\underset{\sim}{a}+\left(\underset{\sim}{d}\cdot\underset{\sim}{e}\right)\underset{\sim}{c}$
3. $\underset{\sim}{a}\underset{\sim}{b}\neq\underset{\sim}{b}\underset{\sim}{a}$

The third property follows easily from the first property.

As noted above, the ***shorthand notation*** for the ***inertia dyadic*** of a body about its mass center G is $\underset{\approx}{I}_G$. This notation is particularly useful when defining the ***angular momentum*** of a body.