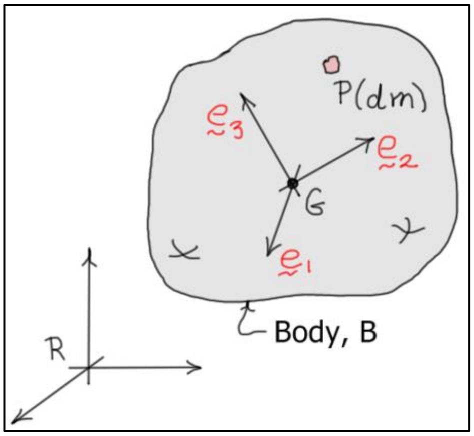
ME 5550 Intermediate Dynamics
Kinetic Energy of a Rigid Body

The figure depicts a rigid body B moving relative to a fixed frame R. The ***kinetic energy*** of B is defined as follows:



$$K = \int_B \tfrac{1}{2}\left({}^R\underset{\sim}{v}_P \cdot {}^R\underset{\sim}{v}_P\right) dm$$

A more ***useful definition*** can be derived by relating the velocity of P to the velocity of the mass center G. Using the relative velocity equation, the integrand can be rewritten as

$$\begin{aligned} {}^R\underset{\sim}{v}_P \cdot {}^R\underset{\sim}{v}_P &= \left({}^R\underset{\sim}{v}_P\right)^2 = \left({}^R\underset{\sim}{v}_G + {}^R\underset{\sim}{v}_{P/G}\right)^2 = \left({}^R\underset{\sim}{v}_G + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right)\right)^2 \\ &= \left({}^R\underset{\sim}{v}_G\right)^2 + 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{r}_{P/G}\right)^2 \end{aligned}$$

Substituting this result into the integral gives the following three integrals:

1. $$\int_B \tfrac{1}{2}\left({}^R\underset{\sim}{v}_G\right)^2 dm = \tfrac{1}{2}\left({}^R\underset{\sim}{v}_G\right)^2 \int_B dm = \tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G\right)^2$$

2. $$\int_B 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right) dm = 2\,{}^R\underset{\sim}{v}_G \cdot \left({}^R\underset{\sim}{\omega}_B \times \left(\int_B \underset{\sim}{r}_{P/G}\, dm\right)\right) = 0$$ (definition of mass center)

3. Setting $\underset{\sim}{r}_{P/G} = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3$ and ${}^R\underset{\sim}{\omega}_B = \omega_1\underset{\sim}{e}_1 + \omega_2\underset{\sim}{e}_2 + \omega_3\underset{\sim}{e}_3$, the integrand of the third integral can be expanded as follows:

$$\begin{aligned} \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)^2 &= (\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2 \\ &= \omega_1^2\left(y^2+z^2\right) + \omega_2^2\left(x^2+z^2\right) + \omega_3^2\left(x^2+y^2\right) \\ &\quad - 2\omega_1\omega_2 xy - 2\omega_1\omega_3 xz - 2\omega_2\omega_3 yz \end{aligned}$$

Substituting into the integral gives

$$\begin{aligned} \int_B \tfrac{1}{2}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/G}\right)^2 dm &= \tfrac{1}{2}\omega_1^2 \int_B \left(y^2+z^2\right) dm + \tfrac{1}{2}\omega_2^2 \int_B \left(x^2+z^2\right) dm + \tfrac{1}{2}\omega_3^2 \int_B \left(x^2+y^2\right) dm \\ &\quad - \omega_1\omega_2 \int_B xy\, dm - \omega_1\omega_3 \int_B xz\, dm - \omega_2\omega_3 \int_B yz\, dm \\ &= \tfrac{1}{2}\omega_1^2 I_{xx}^G + \tfrac{1}{2}\omega_2^2 I_{yy}^G + \tfrac{1}{2}\omega_3^2 I_{zz}^G - \omega_1\omega_2 I_{xy}^G - \omega_1\omega_3 I_{xz}^G - \omega_2\omega_3 I_{yz}^G \\ &= \tfrac{1}{2}\,{}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G \end{aligned}$$

Adding the three results gives the result.

$$\boxed{K = \tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G \right)^2 + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_G = \tfrac{1}{2} m \left({}^R\underset{\sim}{v}_G \right)^2 + \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_B \cdot \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B} \quad \text{(general motion)}$$

Special Case: Motion about a Fixed Point in the Body

If there is a point O within the body that is fixed so that the body pivots about O, then the above result can be simplified as follows.

$$\begin{aligned} \left({}^R\underset{\sim}{v}_G \right)^2 = \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{G/O} \right)^2 &= \left(y_G^2 + z_G^2 \right)\omega_1^2 + \left(x_G^2 + z_G^2 \right)\omega_2^2 + \left(x_G^2 + y_G^2 \right)\omega_3^2 \\ &\quad - 2\omega_1\omega_2 x_G y_G - 2\omega_1\omega_3 x_G z_G - 2\omega_2\omega_3 y_G z_G \end{aligned}$$

Substituting this result into the boxed equation above and combining terms, it can be shown that the kinetic energy can be reduced to the form

$$\boxed{K = \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_O = \tfrac{1}{2}\, {}^R\underset{\sim}{\omega}_B \cdot \underset{\approx}{I}_O \cdot {}^R\underset{\sim}{\omega}_B} \quad \text{(motion about a fixed point in the body)}$$

Here, $\underset{\approx}{I}_O$ is the inertia dyadic (or matrix) for a set of axes through the fixed-point O.