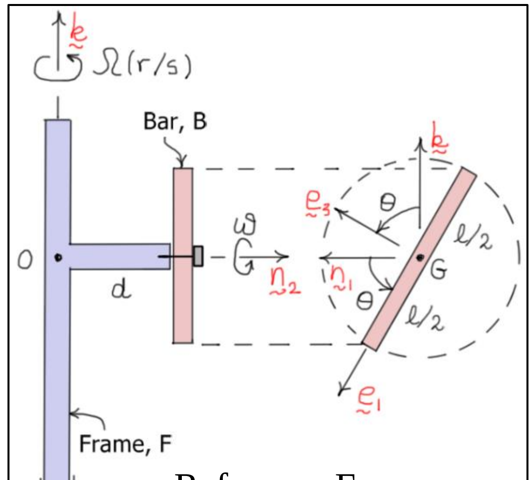
ME 5550 Intermediate Dynamics
Equations of Motion of Example System II

In previous notes for Example System II, ${}^R\underset{\sim}{\omega}_B$ the ***angular velocity*** of the bar, $\left[I_G\right]_{\underset{\sim}{e}}$ the ***inertia matrix*** (associated with $\underset{\approx}{I}_G$) resolved in the ***bar-fixed*** directions $B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$, and $\underset{\sim}{H}_G$ the angular momentum of the bar about its mass-center were found to be



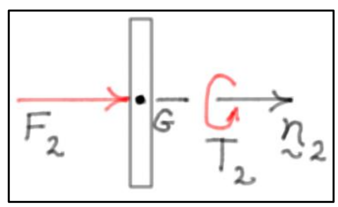
$F:(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{k})$
$B:(\underset{\sim}{e}_1,\underset{\sim}{n}_2,\underset{\sim}{e}_3)$

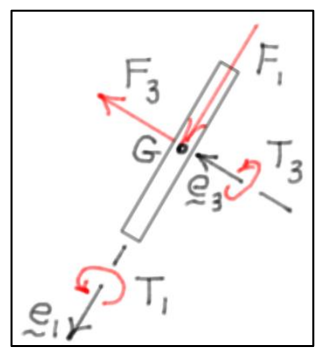
$$\boxed{{}^R\underset{\sim}{\omega}_B = (-\Omega S_\theta)\underset{\sim}{e}_1 + \omega\,\underset{\sim}{n}_2 + (\Omega C_\theta)\underset{\sim}{e}_3}$$

$$\boxed{\left[I_G\right]_{\underset{\sim}{e}} = \frac{m\ell^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{\underset{\sim}{H}_G = \underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\omega}_B = \tfrac{m\ell^2}{12}\left[\omega\,\underset{\sim}{n}_2 + \Omega C_\theta\,\underset{\sim}{e}_3\right]}$$

The equations of motion of B can be found by applying the ***Newton/Euler equations*** to the free-body diagrams shown at the right.





$$\boxed{\begin{aligned} &\sum \underset{\sim}{F} = m\,{}^R\underset{\sim}{a}_G \\ &\sum \underset{\sim}{M}_G = \left(\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G\right) \end{aligned}}$$

Given the angular rate $\Omega = \text{constant}$, the terms on the right side of the moment equation can be calculated as follows.

$${}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G = \frac{m\ell^2}{12}\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{n}_2 & \underset{\sim}{e}_3 \\ -\Omega S_\theta & \omega & \Omega C_\theta \\ 0 & \omega & \Omega C_\theta \end{vmatrix} \;\Rightarrow\; \boxed{{}^R\underset{\sim}{\omega}_B \times \underset{\sim}{H}_G = \frac{m\ell^2}{12}\left(\Omega^2 S_\theta C_\theta\,\underset{\sim}{n}_2 - \omega\Omega S_\theta\,\underset{\sim}{e}_3\right)}$$

$$\left[I_G\right]_{\underset{\sim}{e}}\{\alpha\}_{\underset{\sim}{e}} = \frac{m\ell^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} -\omega\Omega C_\theta \\ \dot{\omega} \\ -\omega\Omega S_\theta \end{Bmatrix} \;\Rightarrow\; \boxed{\underset{\approx}{I}_G \cdot {}^R\underset{\sim}{\alpha}_B = \frac{m\ell^2}{12}\left(\dot{\omega}\,\underset{\sim}{n}_2 - \omega\Omega S_\theta\,\underset{\sim}{e}_3\right)}$$

<u>Inverse Dynamics</u> (assuming Ω and ω are constant)

In this case, forces and torques are calculated to produce the desired motion.

Force Equations:

$$\boxed{\sum \underset{\sim}{F} = F_1 \underset{\sim}{e}_1 + F_2 \underset{\sim}{n}_2 + F_3 \underset{\sim}{e}_3 = m\, {}^R\underset{\sim}{a}_G = m\left(-d\,\Omega^2\, \underset{\sim}{n}_2\right)} \quad \Rightarrow \boxed{\begin{cases} F_1 = F_3 = 0 \\ F_2 = -m\,d\,\Omega^2 \end{cases}} \text{ (inverse dynamics)}$$

Moment Equations:

$$\sum \underset{\sim}{M}_G = T_1 \underset{\sim}{e}_1 + T_2 \underset{\sim}{n}_2 + T_3 \underset{\sim}{e}_3 = \tfrac{1}{12} m\,\ell^2 \left[\left(\dot{\omega} + \Omega^2 S_\theta C_\theta\right)\underset{\sim}{n}_2 - 2\,\omega\Omega S_\theta\, \underset{\sim}{e}_3\right]$$

$$\Rightarrow \boxed{\begin{cases} T_1 = 0 \\ T_2 = \tfrac{1}{12} m\,\ell^2 \left(\dot{\omega} + \Omega^2 S_\theta C_\theta\right) \\ T_3 = -\tfrac{1}{6} m\,\ell^2\, \omega\Omega\, S_\theta \end{cases}} \text{ (inverse dynamics)}$$

<u>Forward Dynamics of Bar *B*</u> (assuming $\Omega = \text{constant}$, $\omega = \dot{\theta}$, and $\dot{\omega} = \ddot{\theta}$)

The same force and moment equations apply as written above. The difference here is that the angular motion of the bar is not constant. Consequently, the force components F_1, F_2, and F_3 and the torque components T_1 and T_3 are as calculated above. The ***moment equation*** about the $\underset{\sim}{n}_2$ direction, however, becomes a ***differential equation*** for ***tracking changes*** in θ. The torque component T_2 can be an ***applied torque*** or it can be a function of the angle θ and its derivatives as with spring and damping effects.

$$\boxed{\begin{array}{l} F_1 = F_3 = 0 \\ F_2 = -md\Omega^2 \end{array}} \qquad \boxed{\begin{array}{c} T_1 = 0 \\ \ddot{\theta} + \Omega^2 S_\theta C_\theta = 12 T_2 / m\ell^2 \\ T_3 = -\tfrac{1}{6} m\ell^2 \omega\Omega S_\theta \end{array}} \qquad \text{(forward bar dynamics)}$$

<u>Equilibrium Positions for the Bar</u>

If torque $T_2(t) \equiv 0$, the bar exhibits ***equilibrium positions***. These positions can be calculated by setting all ***time-varying*** parts of the differential equation to ***zero***. That is,

$$\boxed{\Omega^2 S_\theta C_\theta = 0}$$

This equation is satisfied when $\boxed{\theta = 0, \pi/2}$. The stability of these steady-state positions determines how the bar responds when it is near them. These positions may be ***stable*** or ***unstable***. Generally speaking, if stable, the bar will remain close to the equilibrium position if it is released near it. If the equilibrium position is unstable, the bar will move away from it when released.