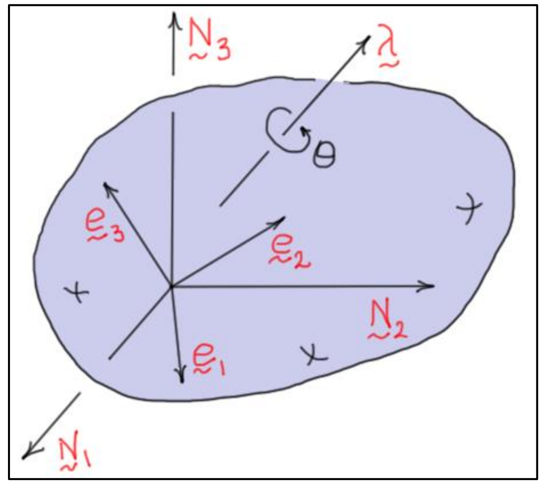
ME 5550 Intermediate Dynamics
Orientation of a Rigid Body Using Euler Parameters

Euler's Theorem on Rotation



Consider the ***rigid body*** shown in the figure. Let $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$ represent the ***base reference frame*** and $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ represent the ***body-fixed reference frame*** and assume ***initially*** that the two frames are ***aligned***. ***Euler's Theorem on Rotation*** states that body $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ can be moved into ***any arbitrary orientation*** relative to the base frame by a rotation about a ***single axis***. In the diagram, θ represents the angle of rotation, and the ***unit vector*** $\underset{\sim}{\lambda}$ represents the direction (or axis) of rotation.

Euler Parameters

The unit vector $\underset{\sim}{\lambda}$ and the angle θ can be related to a set of ***four parameters*** called the ***Euler parameters***. First, let $\underset{\sim}{\lambda}$ be expressed in terms of the base-frame unit vectors as

$$\boxed{\underset{\sim}{\lambda} = \lambda_1 \underset{\sim}{N}_1 + \lambda_2 \underset{\sim}{N}_2 + \lambda_3 \underset{\sim}{N}_3}$$

Then, the four Euler parameters are defined as follows.

$$\boxed{\begin{aligned} \varepsilon_1 &= \lambda_1 \sin(\theta/2) \\ \varepsilon_2 &= \lambda_2 \sin(\theta/2) \\ \varepsilon_3 &= \lambda_3 \sin(\theta/2) \\ \varepsilon_4 &= \cos(\theta/2) \end{aligned}}$$

(four Euler parameters)

Notes

1. The Euler parameters are ***not independent***, because $\boxed{\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2=1}$.

2. It can be shown that the ***unit vectors*** in the two reference frames can be related as follows.

$$\boxed{\begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} = [R] \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}}$$

with

$$\boxed{[R]=\begin{bmatrix} (\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & (-\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2+\varepsilon_4^2) & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & (-\varepsilon_1^2-\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2) \end{bmatrix}}$$

3. If the ***angular velocity*** of the body is resolved into ***components*** in the ***base reference frame***, that is, ${}^R\underset{\sim}{\omega}_B=\omega_1\underset{\sim}{N}_1+\omega_2\underset{\sim}{N}_2+\omega_3\underset{\sim}{N}_3$, then it can also be shown that

$$\boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix} = 2[E] \begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix}} \quad \text{or} \quad \boxed{\begin{Bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{Bmatrix} = \tfrac{1}{2}[E]^T \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{Bmatrix}}$$

with

$$\boxed{[E]=\begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & -\varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix}} \quad \text{and} \quad \boxed{[E]^{-1}=[E]^T}$$

Similar expressions are true for the ***angular velocity components*** about the ***body-fixed axes***. Note that $[E]$ is an ***orthogonal matrix***.

4. Note that ***no singularities*** exist in the kinematic equations shown above, so many computer programs use Euler parameters to ***avoid computational singularities***; however, they may ***communicate*** with the analyst using orientation angles which are easier to visualize.